Correlated Nash Equilibrium

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Abstract

Nash equilibrium presumes that players have expected utility preferences, and therefore the beliefs of each player are represented by a probability measure. Motivated by Ellsberg-type behavior, which contradicts the probabilistic representation of beliefs, we generalize Nash equilibrium in n-player strategic games to allow for preferences conforming to the maxmin expected utility model of Gilboa and Schmeidler [*Journal of Mathematical Economics*, **18** (1989), 141–153]. With no strings attached, our equilibrium concept can be characterized by the suitably modified epistemic conditions for Nash equilibrium.

Keywords: Agreeing to disagree; Correlated equilibrium; Epistemic conditions; Knightian uncertainty; Multiple priors; Nash equilibrium

JEL classification: C72; D81

1 Introduction

It is well known that Nash equilibrium (Nash, 1951) does not allow players' action choices to be correlated; correlated equilibrium (Aumann, 1974), which is a generalization of Nash equilibrium, does. Any correlated equilibrium exhibiting genuine correlation cannot be a Nash equilibrium. So the title of this paper sounds paradoxical. Before explaining this "paradox," we begin with a well-established paradox in decision theory that motivates this paper.

The expected utility model (Savage, 1954) has been the standard representation of preference under uncertainty. A prominent property of expected utility is what Machina and Schmeidler (1992) call *probabilistic sophistication*. In essence, probabilistic sophistication says that a decision maker behaves as if his beliefs are represented by a probability measure. However, the Ellsberg Paradox (Ellsberg, 1961) and related experimental findings (summarized by Camerer and Weber, 1992) demonstrate that when there is ambiguity (about the probability law governing the uncertainty), probabilistic sophistication may be unrealistic; moreover, a decision maker is typically ambiguity averse, which roughly means that

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he prefers to bet on events with known rather than unknown probabilities. Although the Ellsberg Paradox only involves a single decision maker facing an exogenously specified environment, it is natural to think that ambiguity is also common in decision problems where more than one person is involved.

A natural way to accommodate Ellsberg-type behavior is to allow beliefs to be represented by not necessarily one probability measure, but a set of them. In fact, "multiple priors" is a key feature of many generalized expected utility models. Gilboa and Schmeidler's (1989) maxmin expected utility, which will be the focus of this paper, has become a classic example. As its name suggests, the model portrays an ambiguity-averse decision maker who evaluates any act according to its minimum expected utility, where the minimum is taken over the probability measures in his set of priors.

Nash equilibrium has been the central solution concept in game theory. Adopting the modern view that choosing an action in a game is a decision problem under uncertainty, Nash equilibrium presumes that players' preferences are represented by the expected utility model. In order to formally study the implications of ambiguity in strategic situations, we need to address the following central question: How should the definition of Nash equilibrium be generalized to allow for maxmin expected utility preferences? A number of answers have already been given, but none of them is complete. For some of the generalized Nash equilibrium concepts only cover two-player games (e.g., Dow and Werlang (1994), Lo (1999a), Marinacci (2000), Ryan (2002)); some cover n-player games (e.g., Eichberger and Kelsey (2000), Groes et al. (1998), Klibanoff (1996), Lo (1999b)), but formal epistemic foundations of those concepts are not given; some provide equilibrium concepts for n-player games with foundations, but the equilibrium concepts and/or foundations are limited to certain parametric classes of multiple priors, for example, complete ignorance (Lo, 1996), ϵ-contamination (Lo, 2000a), and belief functions (Lo, 2006). Those parametric classes are substantively more restrictive than the intersection of the maxmin expected utility model and Schmeidler's (1989) Choquet expected utility model.[1] But even Choquet expected utility, or its intersection with maxmin expected utility, seems too restrictive. See, for instance, Epstein (1999) and Machina (2007).

This paper provides a complete answer to the question posed in the preceding paragraph. We allow for any number of players, and impose no extraneous restriction whatsoever on multiple priors. With no strings attached, our new generalized Nash equilibrium concept is characterized by—mutual knowledge of rationality, common knowledge of beliefs, and the common prior assumption—the epistemic conditions for Nash equilibrium (Aumann and Brandenburger (1995)), which are suitably modified to allow for ambiguity. The characterization supports the equilibrium concept proposed here as a "minimal" generalization of Nash equilibrium; that is, it deviates from Nash equilibrium *only* in terms of players' attitude towards ambiguity. Consequently, comparing it with Nash equilibrium constitutes a ceteris paribus study of the effects of ambiguity on how a game is played.

To be sure, Lo (1996, 2000a, 2006) emphasizes the importance for a generalized Nash equilibrium concept to possess foundations that are comparable to those of Nash equilibrium. But he also carries the "baggage" that a generalized Nash equilibrium concept must mimic two hallmark features of Nash equilibrium in games with more than two players: agreement

[1] In fact, Dow and Werlang (1994), Eichberger and Kelsey (2000), and Marinacci (2000) adopt the intersection of maxmin expected utility and Choquet expected utility; Groes et al. (1998) adopt belief functions.

and stochastic independence of beliefs. (By agreement of beliefs, we mean that for each player, all the other players hold the same beliefs about the action choice of that player; by stochastic independence of beliefs, we mean that each player believes that the action choices of the other players are stochastically independent.) In this paper, we recognize that it is arguably unnecessary—and even inappropriate—to carry the baggage. As we will explain in detail, agreement and stochastic independence are predictions generated partly by probabilistic sophistication. Suppose that the ultimate objective is to investigate whether ambiguity leads to new predictions on how a game is played. If we start by requiring a generalized Nash equilibrium concept to mimic the predictions of probabilistic sophistication, then the purpose of the exercise will be defeated. In any case, for multiple priors, it is not completely clear how agreement and especially stochastic independence should be defined.

Without the baggage, we are able to see faithfully what the suitably modified epistemic conditions for Nash equilibrium characterize. The resulting generalized Nash equilibrium concept is called *correlated Nash equilibrium*. Correlation is a new prediction. We will even provide an example in which, from each player's perspective, there is ambiguity *only* about how the action choices of his opponents are correlated. As for the issue of agreement, correlated Nash equilibrium implies that players must "partially agree."

The paper proceeds as follows. Section 2 contains a brief review of the maxmin expected utility model, and how it is adapted to the context of a strategic game. Correlated Nash equilibrium is defined in Section 3. Section 4 illustrates, with examples, various properties of correlated Nash equilibrium. The ultimate justification for correlated Nash equilibrium can be found in Section 5. A by-product of the justification is a formalization of "agreeing to partially agree." Section 6 provides weakenings of correlated Nash equilibrium. One of them is a generalization of Dow and Werlang's (1994) equilibrium concept, from two-player games to n-player games, and from the intersection of the maxmin expected utility model and Choquet expected utility model to the maxmin expected utility model. Section 7 concludes.

The following notational conventions will be adopted throughout the paper. For any finite set Q, use $\Delta(Q)$ to denote the set of all probability measures on Q. For any $p \in \Delta(Q)$, let

$$\text{supp } p = \{q \in Q \colon p(q) > 0\}.$$

For any $P \subseteq \Delta(Q)$, let

$$\text{supp } P = \bigcup_{p \in P} \text{supp } p.$$

In words, supp p is the support of the probability measure p, and supp P is the union of the supports of all the probability measures in P.

2 Utility Functions

Let S be a finite set of states, and O a set of outcomes. An act is a function from S to O. Let $\succeq$ be a preference ordering over acts, and $\succ$ the induced strict preference relation. The maxmin expected utility representation for $\succeq$ consists of a von Neuman Morgenstern (vNM) index $u \colon O \to \mathbf{R}$, and a closed and convex set $C \subseteq \Delta(S)$ of probability measures. For any act $f\colon S \to O$, the utility of f is equal to the minimum expected utility

$$\min_{p\in C} \sum_{s\in S} u(f(s))p(s) \tag{1}$$

of f given C. The intuition of Eq. (1) is as follows. The decision maker does not know the probability law governing the state space S. Because of this ambiguity, his beliefs over S are in general represented by a set C of probability measures. Ambiguity aversion is captured by the property that an act is evaluated according to its minimum expected utility, where the minimum is taken over the probability measures in C. Of course, if C is a singleton, then Eq. (1) collapses to expected utility.

In order to isolate the effects of ambiguity aversion, we preserve whatever conventional preference-based notions that are not directly related to attitude towards ambiguity. In particular, following Savage (1954), say that an event $E \subseteq S$ is *nonnull* if there exist acts f and g such that $f(s) = g(s)$ for all states $s \notin E$, and $f \succ g$; otherwise E is *null*. Roughly speaking, E is nonnull (null, respectively) if the decision maker is ever (never, respectively) concerned about what he will receive at states lying inside E. If $\succeq$ is represented by Eq. (1), then E is nonnull if and only if there exists $s \in E$ such that $s \in \operatorname{supp} C$. This justifies the interpretation that the decision maker knows (or "believes with at least probability one") E if $\operatorname{supp} C \subseteq E$.

Let $(A_i, u_i)_{i\in N}$ be a *strategic game*, where $N = \{1, \ldots, n\}$ is a finite set of players, A_i is a finite set of actions (with typical element a_i) available to player i, and $u_i\colon \times_{j\in N} A_j \to \mathbf{R}$ is a vNM index representing i's preference over $\times_{j\in N} A_j$. As usual, let $A = \times_{j\in N} A_j$ be the set of action profiles (with typical element a), and $A_{-i} = \times_{j\neq i} A_j$ be the set of action profiles (with typical element a_{-i}) of players other than player i.

There is no difficulty in adapting the maxmin expected utility model to the context of a strategic game. Since player i is uncertain about the action choices of the other players, the state space for i is A_{-i}. Every action $a_i \in A_i$ can be identified as an act over the state space A_{-i} as follows: If player i chooses the act a_i and the true state is a_{-i}, then i receives the payoff $u_i(a_i, a_{-i})$. Consistent with Eq. (1), player i's beliefs about the action choices of his opponents are represented by a closed and convex set $\Phi_i \subseteq \Delta(A_{-i})$ of probability measures, and given Φ_i, i evaluates a_i according to its minimum expected payoff

$$\min_{\phi_i\in\Phi_i} \sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i})\phi_i(a_{-i}). \tag{2}$$

The central question can now be precisely stated: How should Nash equilibrium be generalized to allow for preferences represented by Eq. (2)? To answer this question, we first present the relevant equilibrium concepts for expected utility preferences.

3 Equilibrium Notions

Although we are *not* providing a generalization of correlated equilibrium, it is nevertheless useful to start with that equilibrium concept. Use σ to denote a probability measure on A, and define,

$$\sigma^{A_i}(a_i) = \sum_{a_{-i}\in A_{-i}} \sigma(a_i, a_{-i}) \quad \forall a_i \in A_i \quad \forall i \in N. \tag{3}$$

In words, $\sigma^{A_i} \in \Delta(A_i)$ is the marginal of σ on A_i. As usual, define

$$\sigma(a_{-i}|a_i) = \frac{\sigma(a_i, a_{-i})}{\sigma^{A_i}(a_i)} \quad \forall a_{-i} \in A_{-i} \quad \forall a_i \in \text{supp } \sigma^{A_i} \quad \forall i \in N. \tag{4}$$

In words, $\sigma(\cdot|a_i) \in \Delta(A_{-i})$ is the conditional of σ given a_i.

Definition 1. A probability measure σ is a *correlated equilibrium* (*distribution*) if it satisfies

$$a_i \in \arg\max_{\hat{a}_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(\hat{a}_i, a_{-i})\sigma(a_{-i}|a_i) \quad \forall a_i \in \text{supp } \sigma^{A_i} \quad \forall i \in N. \tag{5}$$

The classical story behind correlated equilibrium is as follows. A mediator uses the probability law σ to generate confidential recommendations for the players. Every player i knows σ, and if the "message profile" $a \in \text{supp } \sigma$ is actually realized, then player i knows only his component a_i of a. Updating σ using Bayes rule, i's beliefs about the action choices of his opponents are represented by the conditional probability law $\sigma(\cdot|a_i)$. Eq. (5) then says that a_i maximizes i's expected payoff given his beliefs $\sigma(\cdot|a_i)$. Correlated equilibrium gets its name because σ may not be a product measure on A; in other words, (the players believe that) action choices are correlated. While the classical scenario is intuitive, there is no need to "take it literally." Correlated equilibrium can be characterized by common knowledge of rationality and the common prior assumption (Aumann, 1987).

Turn to Nash equilibrium. For reasons that will be apparent shortly, we present the concept in a way that is not the most usual. Parallel to σ^{A_i}, define,

$$\sigma^{A_{-i}}(a_{-i}) = \sum_{a_i \in A_i} \sigma(a_i, a_{-i}) \quad \forall a_{-i} \in A_{-i} \quad \forall i \in N. \tag{6}$$

In words, $\sigma^{A_{-i}} \in \Delta(A_{-i})$ is obtained by marginalizing σ on A_{-i}.

Definition 2. A probability measure σ is a *Nash equilibrium* (*distribution*) if it satisfies Eq. (5), and

$$\sigma(\cdot|a_i) = \sigma^{A_{-i}} \quad \forall a_i \in \text{supp } \sigma^{A_i} \quad \forall i \in N. \tag{7}$$

By Definitions 1 and 2, every Nash equilibrium is a correlated equilibrium. The difference between the two equilibrium concepts is that Nash equilibrium imposes an extra requirement in Eq. (7), which says that every conditional $\sigma(\cdot|a_i)$ is equal to the unconditional $\sigma^{A_{-i}}$; in other words, a_i tells player i nothing about the action choices of his opponents. Eqs. (3), (4) and (7) imply

$$\sigma(a_i, a_{-i}) = \sigma^{A_i}(a_i)\sigma^{A_{-i}}(a_{-i}) \quad \forall a_i \in A_i \quad \forall a_{-i} \in A_{-i} \quad \forall i \in N. \tag{8}$$

Eq. (7), or even Eq. (8), does not (explicitly) say that player i views the action choices of his opponents as stochastically independent. Nevertheless, by Aumann and Brandenburger (1995, p. 1169, Lemma 4.6), if Eq. (8) holds, then the probability law

$$\sigma(a) = \prod_{i \in N} \sigma^{A_i}(a_i) \quad \forall a \in A \tag{9}$$

is a product measure, and so player i's beliefs

$$\sigma(\cdot|a_i) = \prod_{j\neq i} \sigma^{A_j} \tag{10}$$

is a product measure as well. Looking at things this way, Nash equilibrium per se does not require stochastic independence; stochastic independence is an implication of Eq. (7)—an implication one may wish to disentangle.[2] Eq. (10) further implies agreement, in the sense that for each player $i \in N$, all the other players hold the same marginal beliefs σ^{A_i} about i's action choice.

We are ready to propose our equilibrium concept. Adopt Eq. (2) as player i's utility function over A_i, with the associated set $\Phi_i \subseteq \Delta(A_{-i})$ of probability measures representing his beliefs about the action choices of his opponents. From now on, call Φ_i player i's *conjecture*. Use Φ to denote a profile $(\Phi_i)_{i\in N}$ of conjectures.

Definition 3. A pair $\langle \sigma, \Phi \rangle$ is a *correlated Nash equilibrium* if it satisfies

$$\sigma(\cdot|a_i) \in \Phi_i \quad \forall a_i \in \text{supp }\sigma^{A_i} \quad \forall i \in N, \tag{11}$$

$$\text{supp }\Phi_i = \times_{j\neq i}\text{supp }\sigma^{A_j} \quad \forall i \in N, \tag{12}$$

and

$$a_i \in \arg\max_{\hat{a}_i\in A_i} \min_{\phi_i\in\Phi_i} \sum_{a_{-i}\in A_{-i}} u_i(\hat{a}_i, a_{-i})\phi_i(a_{-i}) \quad \forall a_i \in \text{supp }\sigma^{A_i} \quad \forall i \in N. \tag{13}$$

A probability measure σ is a *correlated Nash equilibrium distribution* if there exist a profile Φ of conjectures such that $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium.

If σ is a Nash equilibrium, then $\langle \sigma, (\{\sigma^{A_{-i}}\})_{i\in N} \rangle$ is a correlated Nash equilibrium; conversely, for any correlated Nash equilibrium $\langle \sigma, \Phi \rangle$, if Φ_i is a singleton for all $i \in N$, then $\Phi_i = \{\sigma^{A_{-i}}\}$ for all $i \in N$, and therefore σ is a Nash equilibrium.[3] This confirms that correlated Nash equilibrium is a generalization of Nash equilibrium, *not* correlated equilibrium. However, for an arbitrary correlated Nash equilibrium $\langle \sigma, \Phi \rangle$, σ may not be a product measure, and the probability measures in Φ_i may not be product measures. This explains the word "correlated" in the name of our generalized Nash equilibrium concept.

Let us describe Definition 3 with the following heuristic story, which is modified from the classical story behind correlated equilibrium. Once again, imagine that σ is the probability law used by a mediator to generate confidential recommendations for the players. But now suppose that player i does not know σ, and in particular, he does not know how the mediator's recommendation for him is related to those for the other players. Consequently, when $a \in A$ is the actual message profile generated by σ, and player i knows only his component a_i of a, he is not able to derive the conditional probability law $\sigma(\cdot|a_i)$ governing the action choices of his opponents. This is why his conjecture is represented by a set Φ_i of probability measures.

[2]The desire to disentangle intertemporal substitution from risk aversion (cf. Epstein and Zin, 1989) is, to some extent, an analogy.

[3]Recall that Eq. (7) implies Eq. (10). So if every Φ_i is a singleton, then Eq. (12) is redundant. In general, Eq. (12) does not follow from Eq. (11); also, since σ may not be a product measure, the right-hand side of Eq. (12) may not be the same as supp $\sigma^{A_{-i}}$.

While player i cannot pin down the actual conditional probability law, Eq. (11) says that he is "conservative enough," in the sense that his conjecture contains all the possible conditional probability laws. According to Eq. (12), from player i's point of view, an action profile of his opponents is nonnull if and only if it is contained in $\times_{j \neq i} \text{supp } \sigma^{A_j}$. Thus, for each player $i \in N$, all the other players agree on the set supp σ^{A_i} of actions that will possibly be taken by player i. Eq. (13) then says that for every action a_i that is nonnull from the point of view of i's opponents, a_i gives player i his *correlated Nash equilibrium payoff*, which is his maxmin expected payoff given his conjecture Φ_i.

To provide further perspective, consider briefly the following two superficially plausible alternatives to correlated Nash equilibrium.

Definition 4. A pair $\langle \sigma, \Phi \rangle$ is a *strong correlated Nash equilibrium* if it satisfies Eqs. (9), (11), (12), and (13).[4] A pair $\langle \sigma, \Phi \rangle$ is a *weak correlated Nash equilibrium* if it satisfies Eq. (12), Eq. (13), and

$$\sigma^{A_{-i}} \in \Phi_i \quad \forall i \in N. \tag{14}$$

A probability measure σ is a *strong* (*weak*, respectively) *correlated Nash equilibrium distribution* if there exist a profile Φ of conjectures such that $\langle \sigma, \Phi \rangle$ is a strong (weak, respectively) correlated Nash equilibrium.

As a strong correlated Nash equilibrium satisfies all the conditions in Definition 3, it must be a correlated Nash equilibrium. A correlated Nash equilibrium must be a weak correlated Nash equilibrium. This is because, by Eqs. (4) and (6), the unconditional $\sigma^{A_{-i}}$ is a convex combination of the collection $\{\sigma(\cdot|a_i)\}_{a_i \in \text{supp } \sigma^{A_i}}$ of conditionals. Hence Eq. (11) and convexity of Φ_i imply Eq. (14).

Both the strong, and the weak, correlated Nash equilibrium concepts seem to have some intuition. On the one hand, strong correlated Nash equilibrium captures a stochastically independent probability law σ, whereas correlated Nash equilibrium may not. On the other hand, weak correlated Nash equilibrium appears to be more reasonable than correlated Nash equilibrium, as the former only requires Φ_i to contain the unconditional $\sigma^{A_{-i}}$, rather than the entire collection $\{\sigma(\cdot|a_i)\}_{a_i \in \text{supp } \sigma^{A_i}}$ of conditionals. Nevertheless, if the ultimate objective is to provide an equilibrium concept that is different from Nash equilibrium *only* in terms of players' attitude towards ambiguity, then the correct metric to judge the three equilibrium concepts should be the epistemic conditions for Nash equilibrium. We will prove in Section 5 that, according to this metric, strong correlated Nash equilibrium is too strong, weak correlated Nash equilibrium is too weak, and correlated Nash equilibrium is just right. In fact, it should be apparent that the weak correlated Nash equilibrium concept is too weak. For it does not collapse to Nash equilibrium even when every Φ_i is a singleton.[5]

[4] Equivalently, $\langle \sigma, \Phi \rangle$ is a strong correlated Nash equilibrium if it satisfies Eqs. (9), (12), (13), and (14). Eqs. (9) and (14) together are kind of like Condition (2) in Klibanoff's (1996, p. 8) *equilibrium with uncertainty aversion.*

[5] For an example, see Eq. (32) in the sequel.

	L	R
U	0, 0, 5	0, 0, 0
D	0, 0, 0	0, 0, 0

X

	L	R
U	2, 2, 2	2, 2, 0
D	2, 2, 0	2, 2, 2

Y

	L	R
U	0, 0, 0	0, 0, 0
D	0, 0, 0	0, 0, 5

Z

Figure 1: A three-player game (Example 1).

4 Illustrations

Before proceeding to the foundations of correlated Nash equilibrium, it is useful to get a concrete sense of how the equilibrium concept performs. In this section, we present some properties of correlated Nash equilibrium, and illustrate them with examples. In all the figures, unless specified otherwise, player 1 chooses the row, player 2 the column, and if there is a third player, player 3 the matrix. Payoffs are in terms of vNM utilities.

Say that a correlated Nash equilibrium $\langle \sigma, \Phi \rangle$ is *proper* if at least one Φ_i is not a singleton; put it another way, in a proper correlated Nash equilibrium, at least one player is a genuine maxmin expected utility maximizer. A natural question arises: Why do we have proper correlated Nash equilibria? Proposition 1 below can be regarded as an answer to this question. Loosely speaking, correlation "explains" ambiguity.

Proposition 1. *Suppose that $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium, but not a strong correlated Nash equilibrium. Then $\langle \sigma, \Phi \rangle$ is a proper correlated Nash equilibrium.*

Proposition 1 is true because every correlated Nash equilibrium which is not proper is virtually a Nash equilibrium, and every Nash equilibrium is virtually a strong correlated Nash equilibrium.

Example 1. Consider the game in Figure 1. If player 3 is an expected utility maximizer, he will not choose the action Y. We now construct a correlated Nash equilibrium in which Y is the unique best response for player 3. Let[6]

$$\sigma = (ULY, 1/2; DRY, 1/2) \tag{15}$$

and

$$\Phi_1 = \{\phi_1 \in \Delta(A_{-1})\colon \phi_1(LY) + \phi_1(RY) = 1\} \tag{16}$$
$$\Phi_2 = \{\phi_2 \in \Delta(A_{-2})\colon \phi_2(UY) + \phi_2(DY) = 1\} \tag{17}$$
$$\Phi_3 = \{\phi_3 \in \Delta(A_{-3})\colon \phi_3(UL) + \phi_3(DR) \geq 1/2\}\,. \tag{18}$$

According to Eq. (16), player 1 believes that player 3 will take the action Y, but he is ignorant about what player 2 is going to do. Similarly, according to Eq. (17), player 2 also believes that player 3 will choose Y, but he is ignorant about what player 1 is going to do.

[6] For any finite set Q, $(q_1, p_1; \ldots; q_k, p_k) \in \Delta(Q)$ denotes the probability measure which assigns probability p_1 to $q_1, \ldots, p_k$ to q_k.

	L	R
U	0, 0, 5	0, 0, 0
D	0, 0, 0	0, 0, 5

W

	L	R
U	0, 0, 0	0, 0, 5
D	0, 0, 5	0, 0, 0

X

	L	R
U	2, 2, 2	2, 2, 2
D	2, 2, 0	2, 2, 0

Y

	L	R
U	2, 2, 0	2, 2, 0
D	2, 2, 2	2, 2, 2

Z

Figure 2: A three-player game (Example 2).

As for player 3, he believes that the actions of players 1 and 2 are correlated; to be precise, according to Eq. (18), he assigns probability at least 1/2 to the event that either (U, L) or (D, R) will be chosen by his opponents.

The probability law σ in Eq. (15) delivers the following conditionals:

$$\begin{aligned} &\sigma(\cdot|U) = (LY, 1),\ \ \sigma(\cdot|D) = (RY, 1), \\ &\qquad \sigma(\cdot|L) = (UY, 1),\ \ \sigma(\cdot|R) = (DY, 1),\ \ \sigma(\cdot|Y) = (UL, 1/2; DR, 1/2). \end{aligned} \tag{19}$$

It follows from Eqs. (16)–(19) that $\sigma(\cdot|a_i) \in \Phi_i$ for all $a_i \in \operatorname{supp} \sigma^{A_i}$ and all $i \in N$; that is, σ and Φ satisfy Eq. (11) in the definition of correlated Nash equilibrium. Eqs. (15)–(18) imply $\operatorname{supp} \Phi_i = \times_{j \neq i} \operatorname{supp} \sigma^{A_j}$ for all $i \in N$; that is, σ and Φ satisfy Eq. (12). Given player 3's conjecture Φ_3 as defined in Eq. (18), his minimum expected payoff of taking the action Y is equal to 1, whereas his minimum expected payoff of taking any of the other two actions is equal to 0. So Y is indeed his unique best response given Φ_3. Obviously, player 1 is indifferent between the two actions U and D, and player 2 is indifferent between L and R. Therefore, Eq. (13) is also satisfied. Hence $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium. Note that σ is not a product measure, and $\langle \sigma, \Phi \rangle$ is proper, confirming Proposition 1. □

Example 2. Consider the game in Figure 2. If player 3 is an expected utility maximizer, he will choose neither Y nor Z. But these two actions could be the only best responses for player 3 in a correlated Nash equilibrium. For instance, let

$$\sigma = (ULY, 1/4; DRY, 1/4; URZ, 1/4; DLZ, 1/4) \tag{20}$$

and

$$\Phi_1 = \{\phi_1 \in \Delta(A_{-1})\colon \phi_1(LY) = \phi_1(RZ) \text{ and } \phi_1(RY) = \phi_1(LZ)\} \tag{21}$$

$$\Phi_2 = \{\phi_2 \in \Delta(A_{-2})\colon \phi_2(UY) = \phi_2(DZ) \text{ and } \phi_2(DY) = \phi_2(UZ)\} \tag{22}$$

$$\Phi_3 = \{\phi_3 \in \Delta(A_{-3})\colon \phi_3(UL) = \phi_3(DR) \text{ and } \phi_3(UR) = \phi_3(DL)\}. \tag{23}$$

	L	R
U	$6,6$	$2,7$
D	$7,2$	$0,0$

Figure 3: A two-player game (Example 3).

Player 3's conjecture, for example, can be understood using the following story inspired by Aumann (1987, p. 16). Player 3 believes that players 1 and 2 went to the same business school, but he is ignorant about which one. Nevertheless, he believes that some business schools taught either "UL" or "DR", with equal probabilities; the rest taught "UR" or "DL", with equal probabilities. He also understands that player 1 would take the action U if she learned "UL", and player 2 would take the action L if she learned "UL", etc. This explains Φ_3 as stated in Eq. (23). Note that Φ_3 is not a singleton, but all the probability measures in Φ_3 induce the same marginal $(U, 1/2; D, 1/2)$ on A_1, and the same marginal $(L, 1/2; R, 1/2)$ on A_2. So, from player 3's perspective, there is ambiguity *only* about how the actions of his opponents are correlated.

The probability law σ in Eq. (20) delivers the following conditionals:

$$\sigma(\cdot|U) = (LY, 1/2; RZ, 1/2), \;\; \sigma(\cdot|D) = (RY, 1/2; LZ, 1/2) \tag{24}$$
$$\sigma(\cdot|L) = (UY, 1/2; DZ, 1/2), \;\; \sigma(\cdot|R) = (DY, 1/2; UZ, 1/2) \tag{25}$$
$$\sigma(\cdot|Y) = (UL, 1/2; DR, 1/2), \;\; \sigma(\cdot|Z) = (UR, 1/2; DL, 1/2). \tag{26}$$

Eqs. (21)–(26) imply that $\sigma(\cdot|a_i) \in \Phi_i$ for all $a_i \in \text{supp } \sigma^{A_i}$ and for all $i \in N$. Eqs. (20)–(23) imply supp $\Phi_i = \times_{j \neq i} \text{supp } \sigma^{A_j}$ for all $i \in N$. Given player 3's conjecture Φ_3 as defined in Eq. (23), his minimum expected payoff of Y or Z is equal to 1, whereas his minimum expected payoff of W or X is equal to 0. So Y and Z are indeed the only two best responses given Φ_3. Obviously, player 1 is indifferent between U and D, and player 2 is indifferent between L and R. Hence $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium. Once again, σ is not a product measure, and $\langle \sigma, \Phi \rangle$ is proper. □

Another natural question is: Does the set of correlated Nash equilibria have any structure? Answers are provided in Propositions 2 and 3 below. First, it follows immediately from Definition 3 that, *given* Φ, the set of correlated Nash equilibrium distributions is convex.

Proposition 2. *Suppose that $\langle \sigma, \Phi \rangle$ and $\langle \hat{\sigma}, \Phi \rangle$ are both correlated Nash equilibria. Then for any $\lambda \in [0, 1]$, $\langle \lambda\sigma + (1 - \lambda)\hat{\sigma}, \Phi \rangle$ is also a correlated Nash equilibrium.*

Example 3. Consider the "Chicken" game in Figure 3. In any proper correlated Nash equilibrium $\langle \sigma, \Phi \rangle$, both U and D are best responses for player 1, and both L and R are best responses for player 2; to be precise,

$$\min_{\phi_1 \in \Phi_1} \left[6\phi_1(L) + 2\phi_1(R)\right] = \min_{\phi_1 \in \Phi_1} \left[7\phi_1(L)\right] \tag{27}$$

and

$$\min_{\phi_2\in\Phi_2}\big[6\phi_2(U)+2\phi_2(D)\big]=\min_{\Phi_2\in\Phi_2}\big[7\phi_2(U)\big]. \tag{28}$$

Eqs. (27) and (28) can be simplified to

$$\max_{\phi_1\in\Phi_1}\phi_1(R)=1/3 \quad\text{and}\quad \max_{\phi_2\in\Phi_2}\phi_2(D)=1/3. \tag{29}$$

Let us consider the largest Φ_1 and Φ_2 satisfying Eq. (29); that is, let

$$\Phi_1=\{\phi_1\in\Delta(A_2)\colon \phi_1(R)\le 1/3\} \quad\text{and}\quad \Phi_2=\{\phi_2\in\Delta(A_1)\colon \phi_2(D)\le 1/3\}. \tag{30}$$

Given Φ as specified in Eq. (30), Eqs. (11) and (12) can be reduced to the following system of inequalities:

$$\begin{aligned}\sigma(U,R)>0,\ \sigma(U,L)\ge 2\sigma(U,R)\ge 4\sigma(D,R),\\ \sigma(D,L)>0,\ \sigma(U,L)\ge 2\sigma(D,L)\ge 4\sigma(D,R).\end{aligned} \tag{31}$$

Eq. (31) can be intuitively described as follows. According to any proper correlated Nash equilibrium distribution σ, the event that both players are chicken is at least twice as likely as the event that only player 1 (player 2) is chicken, the event that only player 1 (player 2) is chicken is at least twice as likely as the event that nobody is chicken, and the event that nobody is chicken is the only nonempty event which could happen with zero probability. Note that

$$\sigma=(UL,2/3;DR,1/3),\quad \Phi_1=\{(L,2/3;R,1/3)\},\quad \Phi_2=\{(U,2/3;R,1/3)\} \tag{32}$$

form a weak correlated Nash equilibrium. But the weak correlated Nash equilibrium distribution $(UL,2/3;DR,1/3)$ does not satisfy Eq. (31); the correlated equilibrium distribution $(UL,1/3;UR,1/3;DL,1/3)$ pointed out by Aumann (1974, p. 72) does not satisfy Eq. (31) either. □

In Proposition 2 above, Φ is given while σ varies. Alternatively, we can do the opposite. For any two closed and convex sets $\Phi_i,\hat\Phi_i\subseteq\Delta(A_{-i})$, and any $\lambda_i\in[0,1]$, define the convex combination

$$\lambda_i\Phi_i+(1-\lambda_i)\hat\Phi_i=\left\{\lambda_i\phi_i+(1-\lambda_i)\hat\phi_i\colon \phi_i\in\Phi_i \text{ and } \hat\phi_i\in\hat\Phi_i\right\}.$$

Then it is also an immediate consequence of Definition 3 that, *given* σ, the set of correlated Nash equilibrium conjectures (payoffs) for each player is convex.

Proposition 3. *Suppose that $\langle\sigma,\Phi\rangle$ and $\langle\sigma,\hat\Phi\rangle$ are both correlated Nash equilibria. Then for any $(\lambda_i)_{i\in N}\in[0,1]^n$, $\langle\sigma,(\lambda_i\Phi_i+(1-\lambda_i)\hat\Phi_i)_{i\in N}\rangle$ is also a correlated Nash equilibrium.*

Example 1 revisited. Recall σ and Φ from Eqs. (15)–(18). Define $\hat\Phi_3=\Delta(A_{-3})$. For any $\lambda_3\in[0,1]$, $\langle\sigma,(\Phi_1,\Phi_2,\lambda_3\Phi_3+(1-\lambda_3)\hat\Phi_3)\rangle$ is a correlated Nash equilibrium of the game in Figure 1, with player 3's minimum expected payoff of Y equal to λ_3. □

Turn to welfare analysis. Example 4 below shows that a proper correlated Nash equilibrium could Pareto dominate (both ex ante and ex post) a Nash equilibrium.

Example 4. Consider the game in Figure 4, which is taken from Aumann (1974, p. 69, Example 2.3). In any Nash equilibrium, player 1 chooses D and player 2 chooses L, leading

	L	R
U	0, 8, 0	3, 3, 3
D	1, 1, 1	0, 0, 0

X

	L	R
U	0, 0, 0	3, 3, 3
D	1, 1, 1	8, 0, 0

Y

Figure 4: A three-player game (Example 4).

to the payoff profile $(1, 1, 1)$. There exists a correlated Nash equilibrium in which U is the unique best response for player 1, and R is the unique best response for player 2, leading to the payoff profile $(3, 3, 3)$. For instance, it can be easily verified that the probability law $\sigma = (URX, 1/2; URY, 1/2)$, and conjectures

$$\Phi_1 = \{\phi_1 \in \Delta(A_{-1})\colon \phi_1(RX) + \phi_1(RY) = 1\},$$
$$\Phi_2 = \{\phi_2 \in \Delta(A_{-2})\colon \phi_2(UX) + \phi_2(UY) = 1\}, \ \ \Phi_3 = \{(UR, 1)\}$$

constitute such an equilibrium. Details are omitted. □

However, as a corollary of Proposition 4 below, in any (not necessarily two-player) zero-sum game, a correlated Nash equilibrium can never Pareto dominate a Nash equilibrium.

Proposition 4. *In any correlated Nash equilibrium of any zerosum game, the sum of the correlated Nash equilibrium payoffs of all the players is at most zero.*

The proof of Proposition 4 is as follows. Recall that every correlated Nash equilibrium $\langle \sigma, \Phi \rangle$ satisfies Eqs. (11) and (13). Eq. (11) implies that for all $i \in N$, and all $a_i \in \text{supp}\ \sigma^{A_i}$,

$$\min_{\phi_i \in \Phi_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\phi_i(a_{-i}) \leq \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma(a_{-i}|a_i). \tag{33}$$

According to Eq. (13), the left-hand side of Eq. (33) is player i's correlated Nash equilibrium payoff. The point is that, in any correlated Nash equilibrium of any (not necessarily zerosum) game, player i is never "too optimistic." That is, his correlated Nash equilibrium payoff can never be strictly higher than the right-hand side of Eq. (33), which can be interpreted as his "real" conditional expected payoff. Weighting each side of Eq. (33) with probability $\sigma^{A_i}(a_i)$, and summing over $a_i \in \text{supp}\ \sigma^{A_i}$, we obtain, for all $i \in N$,

$$\sum_{a_i \in \text{supp}\ \sigma^{A_i}} \sigma^{A_i}(a_i) \left[\min_{\phi_i \in \Phi_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\phi_i(a_{-i}) \right] \leq \sum_{a_i \in \text{supp}\ \sigma^{A_i}} \sigma^{A_i}(a_i) \left[\sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})\sigma(a_{-i}|a_i) \right]. \tag{34}$$

Obviously, being a weighted average of the same number, the left-hand side of Eq. (34) is just player i's correlated Nash equilibrium payoff. The right-hand side of Eq. (34) can be simplified to $\sum_{a \in A} u_i(a)\sigma(a)$. Thus, Eq. (34) says that player i's correlated Nash equilibrium

	L	R
U	$1,-3,2$	$1,0,-1$
D	$0,-3,3$	$0,0,0$

X

	L	R
U	$-3,1,2$	$-3,0,3$
D	$0,1,-1$	$0,0,0$

Y

Figure 5: A three-player game (Example 5).

payoff is at most $\sum_{a\in A} u_i(a)\sigma(a)$. So the sum of the correlated Nash equilibrium payoffs of all the players is at most $\sum_{i\in N}\sum_{a\in A} u_i(a)\sigma(a)$. Of course, for any zerosum game, and any $\sigma\in\Delta(A)$, $\sum_{i\in N}\sum_{a\in A} u_i(a)\sigma(a)=0$. This completes the proof.

Proposition 4 has another corollary. Suppose that every player i in a zerosum game has a *nonnegative action* a_i, in the sense that $u_i(a_i,a_{-i})\geq 0$ for all $a_{-i}\in A_{-i}$. Then, in any correlated Nash equilibrium, the correlated Nash equilibrium payoff of each player must be equal to zero.

Finally, we present an important implication of correlated Nash equilibrium, which is an issue only for games with more than two players. For every nonempty proper subset $M\subset N$, define $A_M=\times_{i\in M}A_i$ (with typical element a_M). For every nonempty M with cardinality $|M|\leq n-2$, and every $i\in N\setminus M$, derive

$$\Phi_i^{A_M}=\Bigg\{\phi_i^{A_M}\in\Delta(A_M)\colon \exists\phi_i\in\Phi_i \text{ such that}$$
$$\text{for every } a_M\in A_M, \phi_i^{A_M}(a_M)=\sum_{a_{N\setminus[\{i\}\cup M]}\in A_{N\setminus[\{i\}\cup M]}}\phi_i(a_M,a_{N\setminus[\{i\}\cup M]})\Bigg\}$$

by marginalizing every probability measure in Φ_i on A_M. The set $\Phi_i^{A_M}\subseteq\Delta(A_M)$ represents player i's marginal conjecture about the action choices of the players in M. It is evident that in a correlated Nash equilibrium $\langle\sigma,\Phi\rangle$, the players may not "fully agree." To be precise, player i's marginal conjecture $\Phi_i^{A_M}$ may not be the same as player j's marginal conjecture $\Phi_j^{A_M}$. Nevertheless, the players have to at least "partially agree."

Proposition 5. *Suppose that $\langle\sigma,\Phi\rangle$ is a correlated Nash equilibrium. Then for every nonempty $M\subset N$ with $|M|\leq n-2$, $\cap_{i\in N\setminus M}\Phi_i^{A_M}\neq\emptyset$.*

Correlated Nash equilibrium ensures that, for any subset of players, the marginal conjectures of those players (about the action choices of their common opponents) must be represented by sets of probability measures with a nonempty intersection. To establish Proposition 5, recall that correlated Nash equilibrium satisfies Eq. (11), which implies Eq. (14). Eq. (14) implies $\sigma^{A_M}\in\Phi_i^{A_M}$ for all $i\in N\setminus M$, where σ^{A_M} is obtained by marginalizing σ on A_M.

Example 5. Consider the game in Figure 5. While it is zerosum, player 3 does not have a nonnegative action. So it is not immediately clear what correlated Nash equilibrium payoffs each player can possibly achieve. To find out the answer, first observe that U is a best

	X	Y
U	$1, -1$	$-3, 3$
D	$0, 0$	$0, 0$

Figure 6: A two-player game derived from Figure 5.

response for player 1 given conjecture Φ_1 if and only if

$$\min_{\phi_1^{A_3} \in \Phi_1^{A_3}} \left[\phi_1^{A_3}(X) - 3 \left[1 - \phi_1^{A_3}(X) \right] \right] \geq 0. \tag{35}$$

Similarly, L is a best response for player 2 given conjecture Φ_2 if and only if

$$\min_{\phi_2^{A_3} \in \Phi_2^{A_3}} \left[\left[1 - \phi_2^{A_3}(X) \right] - 3\phi_2^{A_3}(X) \right] \geq 0. \tag{36}$$

However, Proposition 5 tells us that in any correlated Nash equilibrium $\langle \sigma, \Phi \rangle$, we must have $\Phi_1^{A_3} \cap \Phi_2^{A_3} \neq \emptyset$, implying that Eqs. (35) and (36) cannot hold at the same time. This is essentially an application of Billot et al. (2000), who prove that partial agreement and absence of speculation opportunity are equivalent. Because of partial agreement, there is no speculation opportunity for players 1 and 2 on what player 3 is going to do.

We are now able to see that the only possible correlated Nash equilibrium payoff of each player is zero. Without loss of generality, consider any correlated Nash equilibrium in which player 2's unique best response is R; so his correlated Nash equilibrium payoff is zero. Given that 2 chooses R for sure, players 1 (the row player) and 3 (the column player) are virtually playing the game in Figure 6, which is zerosum; moreover, each player has a nonnegative action. Hence their correlated Nash equilibrium payoffs must be equal to zero as well. □

5 Foundations

In this section, we provide epistemic foundations for correlated Nash equilibrium. Let (Ω, μ) be a finite probability space, where Ω is interpreted as a set of states of the world, and μ the probability law governing Ω. Subsets of Ω are called *events*. Suppose that at every state $\omega \in \Omega$, player i does not know μ, and he is averse to ambiguity; as a result, acts on Ω are ranked according to their minimum expected utilities, where the minimum is taken over a closed and convex set $\mathbf{\Pi}_i(\omega) \subseteq \Delta(\Omega)$ of probability measures. To be precise, i's preference at ω is represented by Eq. (1), with Ω in place of S, and $\mathbf{\Pi}_i(\omega)$ in place of C. Call $\mathbf{\Pi}_i(\omega)$ player i's *theory at* ω. We impose two standard assumptions on player i's theories. First, assume that $\omega \in \operatorname{supp} \mathbf{\Pi}_i(\omega)$ for all $\omega \in \Omega$. Second, assume that for every $\omega, \hat{\omega} \in \Omega$, if $\hat{\omega} \in \operatorname{supp} \mathbf{\Pi}_i(\omega)$, then $\mathbf{\Pi}_i(\omega) = \mathbf{\Pi}_i(\hat{\omega})$. Intuitively, the first assumption says that player i always regards the true state as nonnull; the second assumption says that i always knows his actual theory. With these two assumptions, we can define, for every $i \in N$, a function $\mathbf{H}_i \colon \Omega \to 2^{\Omega} \setminus \{\emptyset\}$, with the following three properties: first, the range $\mathbb{H}_i$ of $\mathbf{H}_i$

is a partition of Ω; second, $\mathbf{\Pi}_i \colon \Omega \to 2^{\Delta(\Omega)} \setminus \{\emptyset\}$ is measurable with respect to $\mathbb{H}_i$; third, $\operatorname{supp} \mathbf{\Pi}_i(\omega) = \mathbf{H}_i(\omega)$ for all $\omega \in \Omega$. Call $\mathbf{H}_i(\omega)$ player i's *type at* ω.

For simplicity, let $(A_i, u_i)_{i\in N}$ be the strategic game played at every $\omega \in \Omega$. The action taken by player i at ω is $\mathbf{a}_i(\omega) \in A_i$. Assume that player i knows his own action; that is, $\mathbf{a}_i \colon \Omega \to A_i$ is measurable with respect to $\mathbb{H}_i$. For every $\omega \in \Omega$, use $\mathbf{a}(\omega)$ to denote the action profile $(\mathbf{a}_i(\omega))_{i\in N}$; similarly, for every $i \in N$, $\mathbf{a}_{-i}(\omega)$ denotes $(\mathbf{a}_j(\omega))_{j\neq i}$.

Player i's *conjecture* $\mathbf{\Phi}_i(\omega)$ *at* ω is induced in the natural way from his theory $\mathbf{\Pi}_i(\omega)$ at ω, and the action functions $\mathbf{a}_{-i}$ of his opponents. That is, for every $\omega \in \Omega$,

$$\begin{aligned} \mathbf{\Phi}_i(\omega) = \{\phi_i \in \Delta(A_{-i}) \colon \exists \pi_i \in \mathbf{\Pi}_i(\omega) \text{ such that} \\ \text{for every } a_{-i} \in A_{-i}, \phi_i(a_{-i}) = \pi_i\left(\{\hat{\omega} \in \mathbf{H}_i(\omega) \colon \mathbf{a}_{-i}(\hat{\omega}) = a_{-i}\}\right)\}. \end{aligned} \tag{37}$$

Use $\mathbf{\Phi}(\omega)$ to denote the profile $(\mathbf{\Phi}_i(\omega))_{i\in N}$. For any arbitrary profile Φ of conjectures, define $\|\Phi\|$ to be the event $\{\omega \in \Omega \colon \mathbf{\Phi}(\omega) = \Phi\}$.

Say that player i is *rational at* ω if the action $\mathbf{a}_i(\omega)$ achieves maxmin expected payoff given the conjecture $\mathbf{\Phi}_i(\omega)$; that is,

$$\mathbf{a}_i(\omega) \in \arg\max_{a_i \in A_i} \min_{\phi_i \in \mathbf{\Phi}_i(\omega)} \sum_{a_{-i}\in A_{-i}} u_i(a_i, a_{-i})\phi_i(a_{-i}).$$

Define $\|$rationality$\|$ to be the event $\{\omega \in \Omega \colon$ every player is rational at $\omega\}$.

If $\mathbf{\Pi}_i(\omega)$ is restricted to be a singleton for all ω and all i, then the above framework collapses to the standard one used for studying foundations of game theory (cf. Osborne and Rubinstein, 1994, p. 76). The following definition will also become standard if all theories are singleton sets. Say that μ is a *common prior* if

$$\text{for every } i \in N \text{ and every } \omega \in \Omega, \mu(\mathbf{H}_i(\omega)) > 0 \text{ and } \mu(\cdot|\mathbf{H}_i(\omega)) \in \mathbf{\Pi}_i(\omega). \tag{38}$$

According to Eq. (38), the probability law μ assigns positive probability to every type of every player i, and player i's theory at any state must contain the conditional of μ given his type at that state; intuitively speaking, i's theory "can't go too wrong."

Given the assumptions in the first paragraph of this section, interactive belief is equivalent to interactive knowledge, where the latter is formulated in Aumann (1976).[7] For any $E \subseteq \Omega$, define $K^1(E) = \{\omega \in \Omega \colon \text{For every } i \in N, \mathbf{H}_i(\omega) \subseteq E\}$. The set $K^1(E)$ is interpreted as the set of all states at which every player knows E. If $\omega^* \in K^1(E)$, then say that E is *mutually known at* ω^*. Given $K^1(E)$, recursively define $K^{t+1}(E) = K^1(K^t(E))$ for every positive integer t. If $\omega^* \in \cap_{t=1}^{\infty} K^t(E)$, then say that E is *commonly known at* ω^*. Aumann (1976) proves that an event E is commonly known at ω^* if and only if $\mathbf{H}(\omega^*) \subseteq E$, where $\mathbf{H}(\omega^*)$ is the smallest event such that

$$\omega^* \in \mathbf{H}(\omega^*) \tag{39}$$

and

$$\mathbf{H}_i(\omega) \subseteq \mathbf{H}(\omega^*) \quad \forall \omega \in \mathbf{H}(\omega^*) \quad \forall i \in N. \tag{40}$$

Simply put, the smallest commonly known event at ω^* is the element $\mathbf{H}(\omega^*)$ in the meet of $\{\mathbb{H}_i\}_{i\in N}$ that contains ω^*.

[7] With the common prior assumption, interactive belief and interactive knowledge are also equivalent in the setting of Aumann and Brandenburger (1995). See Lo (2000b) for details.

The ultimate justification for correlated Nash equilibrium is presented in Proposition 6. It shows that correlated Nash equilibrium is the canonical representation of mutual knowledge of rationality, common knowledge of conjectures, and the common prior assumption.

Proposition 6. *Fix a state ω^* and a profile Φ of conjectures. Suppose that $\|rationality\|$ is mutually known and $\|\Phi\|$ is commonly known at ω^*, and μ is a common prior. Then $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium, where*

$$\sigma(a) = \frac{\mu(\{\omega \in \mathbf{H}(\omega^*) \colon \mathbf{a}(\omega) = a\})}{\mu(\mathbf{H}(\omega^*))} \quad \forall a \in A. \tag{41}$$

Conversely, fix a probability measure σ and a profile Φ of conjectures. Suppose that $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium. Then there exists $\langle (\Omega, \mu), (\mathbf{H}_i, \mathbf{\Pi}_i, \mathbf{a}_i)_{i \in N} \rangle$ such that $\|rationality\|$ is mutually known and $\|\Phi\|$ is commonly known at some state ω^*, μ is a common prior, and σ satisfies Eq. (41).

Since σ in Eq. (41) may not be a product measure, the epistemic conditions in Proposition 6 are not sufficient for strong correlated Nash equilibrium. For weak correlated Nash equilibrium, the converse direction of Proposition 6 does not hold. Correlated Nash equilibrium, which is intermediate in strength between strong correlated Nash equilibrium and weak correlated Nash equilibrium, just fits both directions.

Example 1 revisited. We illustrate (the converse direction of) Proposition 6 with the following

- set of states of the world

$$\Omega = \left\{\omega^{ULY}, \omega^{URY}, \omega^{DLY} \omega^{DRY}\right\}. \tag{42}$$

- probability law

$$\mu = \left(\omega^{ULY}, 1/2; \omega^{DRY}, 1/2\right). \tag{43}$$

- information partitions

$$\mathbb{H}_1 = \left\{\{\omega^{ULY}, \omega^{URY}\}, \{\omega^{DLY}, \omega^{DRY}\}\right\} \tag{44}$$

$$\mathbb{H}_2 = \left\{\{\omega^{ULY}, \omega^{DLY}\}, \{\omega^{URY}, \omega^{DRY}\}\right\} \tag{45}$$

$$\mathbb{H}_3 = \left\{\{\omega^{ULY}, \omega^{URY}, \omega^{DLY}, \omega^{DRY}\}\right\}. \tag{46}$$

- theories

$$\mathbf{\Pi}_1\left(\omega^{ULY}\right) = \mathbf{\Pi}_1\left(\omega^{URY}\right) = \left\{\pi_1 \in \Delta(\Omega) \colon \pi_1\left(\omega^{ULY}\right) + \pi_1\left(\omega^{URY}\right) = 1\right\} \tag{47}$$

$$\mathbf{\Pi}_1\left(\omega^{DLY}\right) = \mathbf{\Pi}_1\left(\omega^{DRY}\right) = \left\{\pi_1 \in \Delta(\Omega) \colon \pi_1\left(\omega^{DLY}\right) + \pi_1\left(\omega^{DRY}\right) = 1\right\} \tag{48}$$

$$\mathbf{\Pi}_2\left(\omega^{ULY}\right) = \mathbf{\Pi}_2\left(\omega^{DLY}\right) = \left\{\pi_2 \in \Delta(\Omega) \colon \pi_2\left(\omega^{ULY}\right) + \pi_2\left(\omega^{DLY}\right) = 1\right\} \tag{49}$$

$$\mathbf{\Pi}_2\left(\omega^{URY}\right) = \mathbf{\Pi}_2\left(\omega^{DRY}\right) = \left\{\pi_2 \in \Delta(\Omega) \colon \pi_2\left(\omega^{URY}\right) + \pi_2\left(\omega^{DRY}\right) = 1\right\} \tag{50}$$

$$\mathbf{\Pi}_3\left(\omega\right) = \left\{\pi_3 \in \Delta(\Omega) \colon \pi_3\left(\omega^{ULY}\right) + \pi_3\left(\omega^{DRY}\right) \geq 1/2\right\} \quad \forall \omega \in \Omega. \tag{51}$$

- actions

$$\mathbf{a}_1\left(\omega^{ULY}\right)=\mathbf{a}_1\left(\omega^{URY}\right)=U,\quad \mathbf{a}_1\left(\omega^{DLY}\right)=\mathbf{a}_1\left(\omega^{DRY}\right)=D \tag{52}$$

$$\mathbf{a}_2\left(\omega^{ULY}\right)=\mathbf{a}_2\left(\omega^{DLY}\right)=L,\quad \mathbf{a}_2\left(\omega^{URY}\right)=\mathbf{a}_2\left(\omega^{DRY}\right)=R \tag{53}$$

$$\mathbf{a}_3(\omega)=Y \quad \forall\omega\in\Omega. \tag{54}$$

Eqs. (42)–(51) imply that μ is a common prior. Eqs. (47)–(54) imply $\|\Phi\|=\Omega$, where Φ is the profile of conjectures in Eqs. (16)–(18). Eqs. (44)–(46) imply $\mathbf{H}(\omega^*)=\Omega$ for all $\omega^*\in\Omega$. Eqs. (43) and (52)–(54) imply that if we derive σ from μ according to Eq. (41), with $\mathbf{H}(\omega^*)=\Omega$, then σ in Eq. (15) is obtained. We already know from Example 1 that $\langle\sigma,\Phi\rangle$ is a correlated Nash equilibrium of the game in Figure 1. So if that game is played at every state, then Eq. (13), Eqs. (52)–(54), and $\|\Phi\|=\Omega$ together imply that $\|\text{rationality}\|=\Omega$ as well. To sum up, Eqs. (42)–(54) is a foundation of the correlated Nash equilibrium in Eqs. (15)–(18).

Next, note that $\langle\sigma,\Phi\rangle$, where σ is defined in Eq. (15), $\Phi_1=\{(LY,1/2;RY,1/2)\}$, $\Phi_2=\{(UY,1/2;DY,1/2)\}$, and Φ_3 is defined in Eq. (18), is a weak correlated Nash equilibrium of the game in Figure 1. At first sight, it seems that

$$\mathbf{\Pi}_1\left(\omega^{ULY}\right)=\mathbf{\Pi}_1\left(\omega^{URY}\right)=\left\{\left(\omega^{ULY},1/2;\omega^{URY},1/2\right)\right\} \tag{55}$$

$$\mathbf{\Pi}_1\left(\omega^{DLY}\right)=\mathbf{\Pi}_1\left(\omega^{DRY}\right)=\left\{\left(\omega^{DLY},1/2;\omega^{DRY},1/2\right)\right\} \tag{56}$$

$$\mathbf{\Pi}_2\left(\omega^{ULY}\right)=\mathbf{\Pi}_2\left(\omega^{DLY}\right)=\left\{\left(\omega^{ULY},1/2;\omega^{DLY},1/2\right)\right\} \tag{57}$$

$$\mathbf{\Pi}_2\left(\omega^{URY}\right)=\mathbf{\Pi}_2\left(\omega^{DRY}\right)=\left\{\left(\omega^{URY},1/2;\omega^{DRY},1/2\right)\right\} \tag{58}$$

together with Eqs. (42)–(46) and (51)–(54) above, would be a foundation of this weak correlated Nash equilibrium. However, a more careful reading reveals that this trick does not work. Once Eqs. (47)–(50) are replaced by Eqs. (55)–(58), the probability law μ in Eq. (43) will no longer be a common prior. In fact, the only probability measure that can be a common prior is the uniform probability measure on Ω. But σ in Eq. (15) is not related to this probability measure in the sense of Eq. (41). In fact, it is impossible to construct any $\langle(\Omega,\mu),(\mathbf{H}_i,\mathbf{\Pi}_i,\mathbf{a}_i)_{i\in N}\rangle$ such that the converse direction of Proposition 6 holds for this weak correlated Nash equilibrium. □

In the Appendix, which contains the proof of Proposition 6, we first prove that common knowledge of conjectures and the common prior assumption imply Eq. (11). Recall that, with more than two players, Eq. (11) implies partial agreement. Therefore, common knowledge of conjectures and the common prior assumption imply partial agreement.

Proposition 7. *Fix a state ω^* and a profile Φ of conjectures. Suppose that $\|\Phi\|$ is commonly known at ω^*, and μ is a common prior. Then for every nonempty $M\subset N$ with $|M|\le n-2$, $\cap_{i\in N\setminus M}\Phi_i^{A_M}\neq\emptyset$.*[8]

[8]Proposition 7 supersedes Lo's (2007, p. 127) Proposition 1. In terms of the framework here, Lo essentially assumes that $\|\Phi\|$ is commonly known at ω^*, $\mu(\omega)>0$ for all $\omega\in\Omega$, and for every $i\in N$, there exist two partitions $\mathbb{H}_i$ and $\mathbb{I}_i$ of Ω, such that $\mathbb{I}_i$ is a refinement of $\mathbb{H}_i$, and

$$\mathbf{\Pi}_i(\omega)=\{\pi_i\in\Delta(\Omega)\colon \text{For every } I_i\in\mathbb{I}_i, \pi_i(I_i)=\mu(I_i|\mathbf{H}_i(\omega))\}\quad\forall\omega\in\Omega.$$

The above equation obviously implies that μ is a common prior.

Proposition 7 formalizes "agreeing to partially agree." Since partial agreement is equivalent to absence of speculation opportunity, Proposition 7 can explain casual observations that agents agree to disagree (especially about what other agents are going to do), but do not speculate. For instance, in almost any public discussion of current affairs, the agents involved could easily find at least some disagreement on how a political or economic episode unfold; nevertheless, they may not put their money at stake. It is well known that probabilistic sophistication does not even allow agreeing to disagree (Aumann, 1976). Of course, one could allow agreeing to disagree by dropping the common prior assumption. But agreeing to disagree among expected utility maximizers must lead to speculation. In this sense, those casual observations cannot be explained by standard economic theory.

6 Variations

Let us point out two different ways of defining correlated Nash equilibrium, and if they are adopted, how Proposition 6 should be modified. (Propositions 1, 2, 4 and 5 will continue to be valid.) The first way (in Definition 5 below) is just a minor weakening of Definition 3. In contrast, the second way (in Definition 6 below) is much weaker.

Definition 5. A pair $\langle \sigma, \Phi \rangle$ is a *correlated Nash equilibrium* if it satisfies Eq. (11), and there exist a profile $(S_i)_{i \in N}$ of action sets, where $S_i \subseteq A_i$ for all $i \in N$, such that

$$\text{supp } \Phi_i = \times_{j \neq i} S_j \quad \forall i \in N \tag{59}$$

and

$$a_i \in \arg\max_{\hat{a}_i \in A_i} \min_{\phi_i \in \Phi_i} \sum_{a_{-i} \in A_{-i}} u_i(\hat{a}_i, a_{-i}) \phi_i(a_{-i}) \quad \forall a_i \in S_i \quad \forall i \in N. \tag{60}$$

Clearly, if $S_i = \text{supp } \sigma^{A_i}$ for all $i \in N$, then Definition 5 collapses to Definition 3. (In general, Eqs. (11) and (59) only ensure that supp $\sigma^{A_i} \subseteq S_i$ for all $i \in N$.) Suppose that correlated Nash equilibrium is defined in terms of Definition 5 instead of Definition 3. Then Proposition 6 will go through if $\mu(\mathbf{H}(\omega^*)) > 0$, and Eq. (38) is replaced by

$$\text{for every } i \in N \text{ and every } \omega \in \Omega, \text{if } \mu(\mathbf{H}_i(\omega)) > 0, \text{then } \mu(\cdot | \mathbf{H}_i(\omega)) \in \mathbf{\Pi}_i(\omega). \tag{61}$$

Proposition 7 (which is independent of the definition of correlated Nash equilibrium) will also go through if $\mu(\mathbf{H}(\omega^*)) > 0$, and Eq. (38) is replaced by Eq. (61). Since Eq. (61) does not require μ to assign positive probability to every type of player i, it is weaker than Eq. (38).[9]

Definition 6. A pair $\langle \sigma, \Phi \rangle$ is a *correlated Nash equilibrium* if it satisfies Eqs. (11), (13), and (59).

Definition 6 does not require every action in S_i to be optimal for player i given Φ_i. Therefore, it is weaker than Definition 5. Suppose that correlated Nash equilibrium is defined in terms of Definition 6 instead of Definition 3. Then Proposition 6 will go through if

[9]Kajii and Ui (2006) provides a behavioral characterization of Eq. (61).

$\mu(\mathbf{H}(\omega^*)) > 0$, Eq. (38) is replaced by Eq. (61), and "$\|$rationality$\|$ is mutually known" is replaced by "supp $\mu(\cdot|\mathbf{H}(\omega^*)) \subseteq \|$rationality$\|$".[10] The interpretation of the last condition is that all the players are actually rational at every state that is possible according to $\mu(\cdot|\mathbf{H}(\omega^*))$. However, because mutual knowledge of rationality may not hold, even at those states, a player may not know that his opponents are rational.[11]

Dow and Werlang (1994) consider two-player games and assume that players' preferences are represented by the intersection of the maxmin expected utility model and Choquet expected utility model. They propose an equilibrium concept called *Nash equilibrium under uncertainty*. It can be easily verified that, in terms of maxmin expected utility, (Φ_1, Φ_2) is a Nash equilibrium under uncertainty if and only if there exists σ such that $\langle \sigma, (\Phi_1, \Phi_2) \rangle$ is a correlated Nash equilibrium (Definition 6). So, with foundations, we obtain a generalization of Nash equilibrium under uncertainty, from two-player games to n-player games, and from the intersection of the maxmin expected utility model and Choquet expected utility model to the maxmin expected utility model.

7 Conclusion

In brief: The aim of this paper has been to identify, using *only* the suitably modified epistemic conditions for Nash equilibrium, restrictions of maxmin expected utility preferences in n-player games. While we focus on a single model of preference, the crux of our paper does not hinge on every detail of its functional form. Other "multiple priors models" could potentially be used to formulate similar concepts and results.

Appendix: proof of Proposition 6

Recall the epistemic conditions, which consist of Eq. (38),

$$\mathbf{H}_i(\omega^*) \subseteq \|\text{rationality}\| \quad \forall i \in N, \tag{62}$$

and

$$\Phi_i = \mathbf{\Phi}_i(\omega) \quad \forall \omega \in \mathbf{H}(\omega^*) \quad \forall i \in N, \tag{63}$$

where $\mathbf{H}(\omega^*)$ is the smallest event satisfying Eqs. (39) and (40). We will first prove sufficiency of the epistemic conditions for correlated Nash equilibrium.

Eqs. (38)–(40) imply $\mu(\mathbf{H}(\omega^*)) > 0$. So it is legitimate to define σ as in Eq. (41). We

[10] Since we assume that the same game is played at every state, mutual knowledge of rationality and common knowledge of conjectures together imply common knowledge of rationality (cf. Polak, 1999), which is formally $\mathbf{H}(\omega^*) \subseteq \|$rationality$\|$. Clearly, $\mathbf{H}(\omega^*) \subseteq \|$rationality$\|$ implies supp $\mu(\cdot|\mathbf{H}(\omega^*)) \subseteq \|$rationality$\|$.

[11] Suppose that $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium in the sense of Definition 5. Then every $a_i \in S_i$ is $\mathcal{P}^{MP}$-*rationalizable* in the sense of Epstein (1997, pp. 13–14). It is not the case if correlated Nash equilibrium is defined in terms of Definition 6.

have, for every $i \in N$, every $a_i \in \text{supp}\ \sigma^{A_i}$, and every $a_{-i} \in A_{-i}$,

$$\sigma(a_{-i}|a_i) = \frac{\sigma(a_i, a_{-i})}{\sum\limits_{\hat{a}_{-i} \in A_{-i}} \sigma(a_i, \hat{a}_{-i})} \tag{64}$$

$$= \frac{\mu(\{\omega \in \mathbf{H}(\omega^*) \colon \mathbf{a}_i(\omega) = a_i \text{ and } \mathbf{a}_{-i}(\omega) = a_{-i}\})}{\sum\limits_{\hat{a}_{-i} \in A_{-i}} \mu(\{\omega \in \mathbf{H}(\omega^*) \colon \mathbf{a}_i(\omega) = a_i \text{ and } \mathbf{a}_{-i}(\omega) = \hat{a}_{-i}\})} \tag{65}$$

$$= \frac{\mu(\{\omega \in \mathbf{H}(\omega^*) \colon \mathbf{a}_i(\omega) = a_i \text{ and } \mathbf{a}_{-i}(\omega) = a_{-i}\})}{\mu(\{\omega \in \mathbf{H}(\omega^*) \colon \mathbf{a}_i(\omega) = a_i\})} \tag{66}$$

$$= \frac{\sum\limits_{H_i \in \mathbb{H}_i(a_i)} \mu(\{\omega \in H_i \colon \mathbf{a}_{-i}(\omega) = a_{-i}\})}{\sum\limits_{H_i \in \mathbb{H}_i(a_i)} \mu(H_i)}, \tag{67}$$

where

$$\mathbb{H}_i(a_i) \equiv \{H_i \colon H_i \in \mathbb{H}_i, H_i \subseteq \mathbf{H}(\omega^*), \text{ and for every } \omega \in H_i, \mathbf{a}_i(\omega) = a_i\}. \tag{68}$$

Eq. (41) implies that Eq. (64) can be rewritten as Eq. (65). The equivalence of Eqs. (66) and (67) is due to the measurability of $\mathbf{a}_i \colon \Omega \to A_i$ with respect to $\mathbb{H}_i$. Define, for every $i \in N$, every $a_i \in \text{supp}\ \sigma^{A_i}$, every $H_i \in \mathbb{H}_i(a_i)$, and every $a_{-i} \in A_{-i}$,

$$\phi_i(a_{-i}|H_i) = \frac{\mu(\{\omega \in H_i \colon \mathbf{a}_{-i}(\omega) = a_{-i}\})}{\mu(H_i)}. \tag{69}$$

By Eqs. (37), (38) and (69),

$$\phi_i(\cdot|H_i) \in \mathbf{\Phi}_i(\omega) \quad \forall \omega \in H_i \quad \forall H_i \in \mathbb{H}_i(a_i) \quad \forall a_i \in \text{supp}\ \sigma^{A_i} \quad \forall i \in N. \tag{70}$$

Eqs. (63), (68) and (70) imply

$$\phi_i(\cdot|H_i) \in \Phi_i \quad \forall H_i \in \mathbb{H}_i(a_i) \quad \forall a_i \in \text{supp}\ \sigma^{A_i} \quad \forall i \in N. \tag{71}$$

Use Eq. (69) to rewrite Eq. (67) as

$$\sigma(a_{-i}|a_i) = \frac{\sum\limits_{H_i \in \mathbb{H}_i(a_i)} \mu(H_i)\phi_i(a_{-i}|H_i)}{\sum\limits_{H_i \in \mathbb{H}_i(a_i)} \mu(H_i)}. \tag{72}$$

In words, for every $i \in N$ and every $a_i \in \text{supp}\ \sigma^{A_i}$, $\sigma(\cdot|a_i)$ is a convex combination of $\{\phi_i(\cdot|H_i)\}_{H_i \in \mathbb{H}_i(a_i)}$. Convexity of Φ_i and Eqs. (71)–(72) imply $\sigma(\cdot|a_i) \in \Phi_i$ for all $a_i \in \text{supp}\ \sigma^{A_i}$ and all $i \in N$; that is, σ and Φ satisfy Eq. (11).

Eq. (63) implies

$$\text{supp}\ \Phi_i = \text{supp}\ \mathbf{\Phi}_i(\omega) \quad \forall \omega \in \mathbf{H}(\omega^*) \quad \forall i \in N. \tag{73}$$

Since supp $\mathbf{\Pi}_i(\omega) = \mathbf{H}_i(\omega)$ for all $\omega \in \Omega$ and all $i \in N$, Eqs. (37) and (73) imply

$$\text{supp } \Phi_i = \{\mathbf{a}_{-i}(\hat{\omega})\colon \hat{\omega} \in \mathbf{H}_i(\omega)\} \quad \forall \omega \in \mathbf{H}(\omega^*) \quad \forall i \in N. \tag{74}$$

Eqs. (40) and (74) imply

$$\text{supp } \Phi_i = \{\mathbf{a}_{-i}(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \quad \forall i \in N. \tag{75}$$

Remember that $\mathbf{a}_i\colon \Omega \to A_i$ is measurable with respect to $\mathbb{H}_i$. With this in mind, it is clear that Eqs. (38)–(41) imply

$$\text{supp } \sigma^{A_i} = \{\mathbf{a}_i(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \quad \forall i \in N. \tag{76}$$

Obviously,

$$\{\mathbf{a}_{-i}(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \subseteq \times_{j\neq i}\{\mathbf{a}_j(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \quad \forall i \in N.$$

Let us prove that

$$\{\mathbf{a}_{-i}(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} = \times_{j\neq i}\{\mathbf{a}_j(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \quad \forall i \in N. \tag{77}$$

If $n = 2$, the two sides of Eq. (77) are of course identical. Assume $n > 2$. Without loss of generality, suppose there exists $a_{-1} \in A_{-1}$ such that

$$a_{-1} \in \times_{j=2}^{n}\{\mathbf{a}_j(\omega)\colon \omega \in \mathbf{H}(\omega^*)\} \tag{78}$$

and

$$a_{-1} \notin \{\mathbf{a}_{-1}(\omega)\colon \omega \in \mathbf{H}(\omega^*)\}. \tag{79}$$

Eqs. (78) and (79) imply that for every $\omega \in \mathbf{H}(\omega^*)$ with $\mathbf{a}_2(\omega) = a_2$, there exists $j \in \{3, \ldots, n\}$ such that $\mathbf{a}_j(\omega) \neq a_j$. Combining this with Eqs. (40) and (74), as well as the measurability of $\mathbf{a}_2\colon \Omega \to A_2$ with respect to $\mathbb{H}_2$, we have

$$\text{for each } \omega \in \mathbf{H}(\omega^*), \text{ there exists } j \in \{3, \ldots, n\} \text{ such that } \mathbf{a}_j(\omega) \neq a_j. \tag{80}$$

If $n = 3$, Eq. (80) is equivalent to $a_3 \notin \{\mathbf{a}_3(\omega)\colon \omega \in \mathbf{H}(\omega^*)\}$, contradicting Eq. (78). If $n > 3$, then Eqs. (78) and (80) imply that for each $\omega \in \mathbf{H}(\omega^*)$ with $\mathbf{a}_3(\omega) = a_3$, there exists $j \in \{4, \ldots, n\}$ such that $\mathbf{a}_j(\omega) \neq a_j$. Combining this with Eqs. (40) and (74), as well as the measurability of $\mathbf{a}_3\colon \Omega \to A_3$ with respect to $\mathbb{H}_3$, we have, for each $\omega \in \mathbf{H}(\omega^*)$, there exists $j \in \{4, \ldots, n\}$ such that $\mathbf{a}_j(\omega) \neq a_j$. Repeat this argument as many times as necessary to arrive at $a_n \notin \{\mathbf{a}_n(\omega)\colon \omega \in \mathbf{H}(\omega^*)\}$, contradicting Eq. (78). This completes the proof of Eq. (77). Eqs. (75)–(77) imply supp $\Phi_i = \times_{j\neq i}\text{supp } \sigma^{A_j}$ for all $i \in N$; that is, σ and Φ satisfy Eq. (12).

Eqs. (62)–(63) imply

$$\mathbf{H}_j(\omega^*) \subseteq \|\text{rationality}\| \cap \{\omega \in \Omega\colon \Phi_i = \mathbf{\Phi}_i(\omega)\} \quad \forall i \in N \quad \forall j \neq i. \tag{81}$$

Eqs. (12) and (74) imply

$$\text{supp } \sigma^{A_i} = \{\mathbf{a}_i(\omega)\colon \omega \in \mathbf{H}_j(\omega^*)\} \quad \forall i \in N \quad \forall j \neq i. \tag{82}$$

Eqs. (81) and (82) imply that every $a_i \in \text{supp } \sigma^{A_i}$ maximizes player i's maxmin expected payoff given conjecture Φ_i; that is, σ and Φ satisfy Eq. (13). This completes the proof that $\langle \sigma, \Phi \rangle$ is a correlated Nash equilibrium.

Turn to the converse direction. Fix any correlated Nash equilibrium $\langle \sigma, \Phi \rangle$. Define

$$\Omega = \times_{i \in N} \text{supp } \sigma^{A_i}, \tag{83}$$

and define, for every $a \in \Omega$ (and every $i \in N$),

$$\mu(a) = \sigma(a), \tag{84}$$

$$\mathbf{H}_i(a) = \{a_i\} \times \times_{j \neq i} \text{supp } \sigma^{A_j}, \tag{85}$$

$$\begin{aligned} \mathbf{\Pi}_i(a) = \big\{ & \pi_i \in \Delta(\Omega) \colon \exists \phi_i \in \Phi_i \text{ such that} \\ & \text{for every } \hat{a}_{-i} \in \times_{j \neq i} \text{supp } \sigma^{A_j}, \phi_i(\hat{a}_{-i}) = \pi_i(a_i, \hat{a}_{-i}) \big\}, \end{aligned} \tag{86}$$

and

$$\mathbf{a}_i(a) = a_i. \tag{87}$$

Eqs. (11) and (12) ensure that Eq. (84) and (86) are well defined. Eqs. (86) and (87) imply

$$\|\Phi\| = \Omega. \tag{88}$$

Eqs. (13), (83), (87) and (88) imply $\|\text{rationality}\| = \Omega$. Eqs. (83)–(85) imply

$$\mu(a_i, \hat{a}_{-i} | \mathbf{H}_i(a)) = \sigma(\hat{a}_{-i} | a_i) \quad \forall \hat{a}_{-i} \in \times_{j \neq i} \text{supp } \sigma^{A_j} \quad \forall a \in \Omega \quad \forall i \in N. \tag{89}$$

Eqs. (11), (86) and (89) imply $\mu(\cdot | \mathbf{H}_i(a)) \in \mathbf{\Pi}_i(a)$ for all $a \in \Omega$ and all $i \in N$. Eqs. (83) and (85) imply $\mathbf{H}(a) = \Omega$ for all $a \in \Omega$. Eq. (84) implies that σ satisfies Eq. (41), with $\mathbf{H}(\omega^*) = \Omega$.

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