First-Price, Second-Price, and English Auctions with Resale

2008

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Abstract

In the independent-private-value model, we allow resale among bidders following a first-price sealed-bid, second-price sealed-bid, or English auction with two bidders. We consider two regimes with regard to the disclosure of the sealed bids: full disclosure and no disclosure. Either the auction winner or the auction loser chooses the resale mechanism. Thanks to three key properties our model shares with the common-value model, we obtain explicit formulas for the equilibria. We circumvent the "ratchet effect," by "randomizing" every pure equilibrium under no disclosure into an equivalent behavioral equilibrium under full disclosure. We compare the auctioneer's revenues across auctions and bargaining procedures. We present some n-bidder extensions.

1. Introduction

Because resale is at least possible after many real-life auctions, we add to the independent-private-value model a post-auction stage where resale under incomplete information may take place between bidders. We consider two

[1] I am grateful to the editor and the two referees for their very judicious and constructive comments. I thank Rod Garratt, Thomas Tröger, and Charles Zheng for helpful discussions. I also thank the participants to the 2005 Econometric Society World Congress (London), the 2007 North American Winter Meeting of the Econometric Society (Chicago), the 2007 Conference on Auctions with Resale (UCSB), the 2007 Canadian Economic Theory Conference (Montreal), the 2007 Canadian Economic Association Meetings (Halifax), and the seminars of the departments of economics at the University of Iowa and Brock University. The Social Sciences and Humanities Research Council of Canada provided financial support for this research.

regimes regarding the disclosure of sealed bids: full disclosure, or FD, where the auctioneer reveals all bids before resale, and no disclosure, of ND, where the auctioneer defers the publication of the bids and the auction payments, which may be linked to the bids, until after resale. Although unrealistic for the second-price auction, or SPA, the ND regime is worth studying because it provides equilibria that can be transformed into equilibria under the FD regime. The auction winner in the general case or the only loser when there are two bidders chooses and implements a secondary auction. The bidders are risk-neutral and do not discount their future payoffs[2].

Under FD, bidders engage in signalling through their bids. Because of a "ratchet effect," no perfect Bayesian equilibrium, or PBE, where the bidders follow different strictly increasing and differentiable bidding functions exists when there are two bidders. Consequently, no fully-separating PBE exists in the first price auction, or FPA, with exante heterogeneous bidders, since only different bidding functions could possibly satisfy the bidders'–different–first-order conditions, or FOC's. In the SPA with arbitrary bidders, the only separating PBE is the only symmetric separating PBE: the truth-bidding PBE; although, because of the possibility of resale, it is not an equilibrium in weakly dominant strategies (on this point, see, for example, Gupta and Lebrun, 1999).

Under ND, because bidders cannot directly signal through their bids[3], there exist one pure separating PBE of the FPA and a multiplicity of such equilibria of the SPA. We single out the three properties of our model that allow to explicitly characterize the PBE's through methods from the literature on the common-value model .

Through a "randomization procedure," we transform every PBE under ND into an equivalent semi-separating PBE under FD. The procedure consists in having the resale-price taker along the equilibrium path, that is, the

[2]The results can be extended to the case of arbitrary common discounting with no consumption between auction and resale (see Footnote 6).

[3]They can signal only indirectly, through winning or losing the auction.

less (more) aggressive bidder when the auction winner (loser) sets the same resale price, randomize over a range of bids in such a way that the other bidder sets the resale price he would have chosen had the bids stayed secret. It produces strategies to which again the methods from the common-value model apply. There exists no other PBE's that satisfy some assumptions of monotonicity, "full support," and differentiability.

With two exante homogeneous bidders, the PBE of the FPA and the truth-bidding PBE of the SPA bring the same revenues, which are higher than the revenues from any other PBE's of the SPA. With two exante heterogeneous bidders, some PBE's of the SPA give higher revenues, and some others lower revenues.

Changing the resale-price maker from the auction loser to the auction winner increases the auctioneer's expected revenues. Under some assumptions, the same change does not affect the range of equilibrium revenues from the SPA.

We extend our results to a family of hybrid auctions, the (k_1, k_2)-price auctions.

With n bidders and the auction winner who chooses the resale mechanism, we construct PBE's of the English auction, or EA, by having bidders in the stages leading to the last stage drop when the price reaches their values and having the two bidders remaining at the last stage follow one of the PBE's of the SPA with resale between them.

For the FPA with n bidders, the FOC's and boundary conditions provide an implicit characterization that is amenable to the methods developed for the FPA with no resale and heterogenous bidders. We prove the existence and uniqueness of a differentiable FPA in a "partial-disclosure" regime when there are one "strong" bidder and $n-1$ "weak" and homogeneous bidders. The strong bidder's bidding function is smaller and his bid distribution is larger. The randomization procedure produces a behavioral PBE of the FPA under FD where the strong bidder randomizes over a range of bids.

We finally compare the PBE of the FPA with the PBE's of the EA.

2. The ND Regime

2.1 Theorem 1

One item is being auctioned to one of two risk-neutral bidders, bidder 1 and bidder 2, through a sealed-bid FPA or SPA with reserve price c, mandatory participation, and the fair tie-breaking rule[4]. At the unique resale stage[5], which immediately follows the auction[6], either the auction winner or the auction loser makes a take-it-or-leave-it offer to the other bidder. We first assume, in this section and the next, that the auction winner proposes the resale price. Only the identity of the winner is made public after the auction and before resale.

The bidders' values for their own consumptions of the item are private and independently distributed over the interval[7] $[c, d]$, with $c < d$, according to absolutely continuous probability measures F_1 and F_2 with density functions f_1 and f_2 that are strictly positive and continuous[8]. We use the same notations F_1 and F_2 for the cumulative distribution functions and assume that the (buyer-) virtual-value functions $v - \frac{1-F_1(v)}{f_1(v)}$ and $v - \frac{1-F_2(v)}{f_2(v)}$ are strictly increasing.

We call the (seller's) optimal-resale-price function ρ^s the function whose value at (w_1, w_2) in $[c, d]^2$ is the resale price that maximizes the expected

[4] Our equilibria remain equilibria under voluntary participation and any tie-breaking rule. The results about the SPA easily extend to an arbitrary binding reserve price.

[5] Alternatively, resale may occur at further stages and the resale-price maker has the ability to commit. A take-it-or-leave-it offer as in the text is then the resale-price maker's optimal "transparent" mechanism (see Section 7).

[6] Or, equivalently, the bidders use the same discount factor $\delta = 1$. If, instead, bidders use the same discount factor $\delta < 1$ and resale and consumption occur T periods after the auction rather than immediately following it, the results go through by multiplying all bids by δ^T.

[7] The results straightforwardly extend to value intervals with different upper extremities. See Footnotes 18 and 32.

[8] For many results, these assumptions too can be loosened, for example, to allow density functions that are defined and strictly positive only over $(c, d]$ (as long as they are bounded).

payoff of bidder k with value w_k when he initially owns the item and bidder l's value is distributed according to the restriction of F_l to $[c, w_l]$, where w_k and w_l are the lower and higher components of (w_1, w_2), that is, $\rho^s(w_1, w_2)$ is the solution to the equation below:

$$w_k = \rho^s(w_1, w_2) - \frac{F_l(w_l) - F_l(\rho^s(w_1, w_2))}{f_l(\rho^s(w_1, w_2))}, \ (1)$$

with k and l such that $\{k, l\} = \{1, 2\}$ and $w_k \leq w_l$. That equation (1) has a unique solution follows from the continuity and strict monotonicity of the virtual-value functions. We denote ρ_i^s the function ρ^s with bidder i's value as the first argument, that is: $\rho_1^s(v, w) = \rho_2^s(w, v) = \rho^s(v, w)$.

A strategy of bidder i includes a bidding strategy $\beta_i(.)$–a (measurable) bidding function, if the strategy is pure–and a (measurable) resale-price function $\gamma_i(.;.)$. If bidder i with value v_i follows (β_i, γ_i), he bids $\beta_i(v_i)$ at auction and makes at resale the take-it-or-leave-it offer at the price $\gamma_i(v_i; b_i)$ when he has won the auction with the bid b_i. We assume that a bidder accepts a resale price if and only if it is not larger than his value. We call PBE any couple of strategies $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ that can be completed into a PBE[9]. When β_i is a strictly increasing bidding function, we take the value of its inverse, which we denote α_i, to be d over bids above its range.

In Theorem 1 below, for all $i \neq j$, when bidder j follows a strictly increasing and continuous bidding function such that $\beta_j(c) = c$, bidder i follows, after winning, the following resale-price function:

$$\gamma_i(v; b) = \rho_i^s(v, \max(v, \alpha_j(b))), \ (2)$$

for all (v, b) in $[c, d] \times [c, +\infty)$. From the definition of ρ_i^s, $\gamma_i(v; b)$ is optimal for bidder i according to his revised beliefs, which the conditional of F_j on $[c, \alpha_j(b)]$ represents.

[9] By adding beliefs as functions of the past observed histories.

Hafalir and Krishna (2008) prove[10] Theorem 1 (ii). Our methods of proof, which we develop in the next section, work across auction procedures and disclosure regimes and hence provide an alternative proof to this part of the theorem.

Theorem 1:

(i) SPA: *If* φ *be a strictly increasing continuous function over* $[c,d]$ *such that* $\varphi(c) = c$ *and* $\varphi(d) = d$ *and*

$$\begin{aligned} \beta_1(v) &= \rho^s(v, \varphi(v)), \\ \beta_2(v) &= \rho^s\left(\varphi^{-1}(v), v\right), \end{aligned}$$

for all v *in* $[c,d]$*, and* γ_1, γ_2 *are as in (2), then* $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ *is a PBE of the SPA under ND and the following equality holds true:*

$$\alpha_2 \beta_1 = \varphi.$$

(ii) FPA: *If*

$$\beta_i(v) = \frac{\int_0^{F_i(v)} \rho^s\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq}{F_i(v)}, (3)$$

$i = 1, 2$*, and* γ_1, γ_2 *are as in (2), then* $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ *is a PBE of the FPA under ND and the following equality holds true:*

$$\alpha_2 \beta_1 = F_2^{-1} F_1 . (4)$$

Moreover, β_1, β_2 *are the unique bidding functions part of a pure PBE that are, over* $(c,d]$*, strictly increasing, differentiable, and such that* $\beta_1(c) = \beta_2(c) = c$*.*

[10] They even prove the uniqueness among all pure and nondecreasing PBE's.

From Gupta and Lebrun (1999), the equilibrium bidding functions (3) in the FPA are the same as in the simple model where private information becomes public before resale and the resale price is exogenously determined from the values according to ρ^s. Also from Gupta and Lebrun (1999), the same bid distributions arise at the equilibrium of the symmetric model where both bidders' values are distributed according to G^s such that $(G^s)^{-1}(q) = \rho^s\left(F_1^{-1}(q), F_2^{-1}(q)\right)$, for all q in $[0,1]$.

We will illustrate most of our results with the two examples below, where $\varphi = \alpha_2\beta_1$.

Example 1-Bidder 1 is everywhere more aggressive (Figure 1): $\beta_1(v) > \beta_2(v)$ and hence $\varphi(v) > v$, for all v in (c,d). In the FPA, this is the case if and only if $F_1(v) > F_2(v)$ over (c,d), that is, when F_2 first-order (strictly) stochastically dominates F_1.

Example 2-The bidding functions cross once (Figure 2): $\beta_1(v) > \beta_2(v)$ and $\varphi(v) > v$, for all v in (c,z); $\beta_1(z) = \beta_2(z)$ and $\varphi(z) = z$; and $\beta_1(v) < \beta_2(v)$ and $\varphi(v) < v$, for all v in (z,d). This is the case in the FPA if, for example, F_2 second-order stochastically dominates F_1.

2.2 Proof of Theorem 1

Assume β_1, β_2 are the strictly increasing and continuous bidding functions the bidders are expected to follow. Bidder i derives some expected utility $u_i^{s,w}$ (gross of the auction price) from winning and some utility $u_i^{s,l}$ from losing. Since bidder j, $j \neq i$, does not observe b_i when he makes an offer at resale, bidder i's utility $u_i^{s,l}$ from losing does not depend on b_i and we may, when looking for equilibria, focus on bidder i's net expected utility $\int_c^{\alpha_j(b_i)} u_i^s\left(v_i, v_j; b_i, \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j)$ with respect to the status quo of his losing with probability one, where his net-value function u_i^s–the difference between $u_i^{s,w}$ and $u_i^{s,l}$–is as follows (with v_i as the first argument of u_i^s):

$$
\begin{aligned}
& u_i^s\left(v_i, v_j; b_i, b_j; \beta_i, \beta_j\right) \\
= \; & \rho_i^s\left(v_i, \max\left(v_i, \alpha_j\left(b_i\right)\right)\right), \text{ if not larger than } v_j; (5) \\
= \; & \rho_j^s\left(v_j, \max\left(v_j, \alpha_i\left(b_j\right)\right)\right), \text{ if not larger than } v_i; (6) \\
= \; & v_i, \text{ otherwise.} (7)
\end{aligned}
$$

When resale could take place at the price one of the two bidders would offer, bidder i's net value is equal to the resale price. Otherwise, bidder i's net value is, as when resale is forbidden, equal to his value. The net values are endogenous since they depend on the inverses of the bidding functions β_1, β_2.

Because bidder i's bid b_i can enter his net value only as an argument of his resale price, which he chooses optimally, $b_i' = b_i$ is a solution of the maximization problem below:

$$
b_i \in \arg\max_{b_i' \geq c} \int_c^{\alpha_j(b_i)} u_i^s\left(v_i, v_j; b_i', \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right). \quad (8)
$$

By an envelope theorem (see Appendix 1), we find Lemma 1 (i) below, which is the formal expression of the lack of a first-order effect of a bid change on the expected payoff through the resale price and allows us to focus on the net value for identical bids. Lemma 1 (ii) comes from the correction by resale of any inefficient allocation after a tie, which follows from the obvious inequality $\rho^s\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right) \leq \max\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right)$. The monotonicity in Lemma 1 (iii)[11] comes from the monotonicity of the reseller's optimal resale price with respect to his value.

Lemma 1:

(i) (no direct first-order effect of own bid) For all (v_i, b_i) *in* $[c, d] \times$

[11] The continuity in (iii) is not necessary to prove Theorem 1 (i).

$[c, +\infty)$,

$$\int_c^{\alpha_j(b_i)} u_i^s\left(v_i, v_j; b_i, \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j)$$
$$= \int_c^{\alpha_j(b_i)} u_i^s\left(v_i, v_j; \beta_j(v_j), \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j).$$

(ii) (common net value for common bid) For all $b \geq c$,

$$u_1^s(\alpha_1(b), \alpha_2(b); b, b; \beta_1, \beta_2) = u_2^s(\alpha_2(b), \alpha_1(b); b, b; \beta_1, \beta_2) = \rho^s(\alpha_1(b), \alpha_2(b)).$$

(iii) (monotonicity and continuity) $u_i^s\left(v_i, \alpha_j(b); b, b; \beta_i, \beta_j\right)$ is non-decreasing with respect to v_i in $[c, d]$ and continuous with respect to $b \geq c$.

Proof: See Appendix 1.

To simplify the notation, we drop β_1, β_2 from the arguments of u_i^s. Theorem 1 follows easily from the properties in Lemma 1. Indeed, from Lemma 1 (i), bidder i's expected net payoffs (net of the auction price) when his value is v_i and his bid is b are as follows.

$$\text{SPA:} \quad \int_c^{\alpha_j(b)} \left(u_i^s\left(v_i, v_j; \beta_j(v_j), \beta_j(v_j)\right) - \beta_j(v_j)\right) dF_j(v_j); \quad (9)$$
$$\text{FPA}: \quad \int_c^{\alpha_j(b)} u_i^s\left(v_i, v_j; \beta_j(v_j), \beta_j(v_j)\right) dF_j(v_j) - \int_c^{\alpha_j(b)} b\,dF_j(v_j). \quad (10)$$

Since, at an equilibrium, b should be optimal if $v_i = \alpha_i(b)$, we obtain from

Lemma 1 (ii) and (iii) the FOC's (11) and (12)[12] below:

$$\text{SPA:} \qquad \rho^s\left(\alpha_1(b), \alpha_2(b)\right) = b. \ (11)$$
$$\text{FPA} \ : \ \frac{d}{db}\ln F_i\left(\alpha_i(b)\right) = \frac{1}{\rho^s\left(\alpha_1(b), \alpha_2(b)\right) - b}, \ i = 1, 2. \ (12)$$

The bidding functions in Theorem 1 (i) satisfy (11) and, from Gupta and Lebrun (1999), those in Theorem 1 (ii) form the unique solution to the two conditions (12)[13]. These FOC's together with the "second-order" condition–Lemma 3 (iii)–imply that any bidder's expected net payoff reaches its maximum at the bid his bidding function in Theorem 1 specifies. Since $\beta_j(v_j)$ is equal to $u_i^s\left(\alpha_i\left(\beta_j(v_j)\right), v_j; \beta_j(v_j), \beta_j(v_j)\right)$, the integrand in (9) is nonnegative when $\alpha_j\beta_i(v_i) > v_j$, nonpositive otherwise, and the integral is maximized at $b = \beta_i(v_i)$. For $b \leq \beta_i(v_i)$, the derivative $\left(u_i^s\left(v_i, \alpha_j(b); b, b\right) - b\right)\frac{d}{db}F_j\left(\alpha_j(b)\right) - F_j\left(\alpha_j(b)\right)$ of (10) is not smaller than its value–zero–at $v_i = \alpha_i(b)$. Similarly, the derivative is nonpositive for $b \geq \beta_i(v_i)$. Theorem 1 is proved.

2.3. Properties of the PBE's

Assume that, in the PBE $(\beta_1, \gamma_1; \beta_2, \gamma_2)$, bidder i is less aggressive at his value v, that is, $\beta_i(v) < \beta_j(v)$. Let $[\varphi^-(v), \varphi^+(v)]$ be the maximum interval including v such that bidder i is less aggressive everywhere in its interior. By continuity, the bidding functions coincide at the extremities and we have:

$$\begin{aligned} \varphi^-(v) &= \max\{w \in [c, v] \,|\, \varphi(w) = w\} \\ \varphi^+(v) &= \min\{w \in [v, d] \,|\, \varphi(w) = w\}, \end{aligned}$$

[12] The same FOC's (12) for the FPA would follow from any other choice of optimal resale-offer functions. If a resale-offer function ρ_i is optimal, it must satisfy $\rho_i\left(\alpha_i(b), \alpha_j(b)\right) = \rho_i^s\left(\alpha_i(b), \alpha_j(b)\right)$ if $\alpha_i(b) < \alpha_j(b)$. Along a PBE, if $\alpha_i(b) = \alpha_j(b)$, whether resale occurs or not, both bidders' net values for winning are equal to $\alpha_i(b) = \alpha_j(b)$, which is also equal to $\rho_i^s\left(\alpha_i(b), \alpha_j(b)\right)$.

[13] With the immediate boundary condition $\beta_1(d) = \beta_2(d)$.

where $\varphi = \alpha_2 \beta_1$. Everywhere in the interior of the bid interval $[\beta_i(\varphi^-(v)), \beta_i(\varphi^+(v))]$, we have $\alpha_i(b) > \alpha_j(b)$.

Denote r the function $\rho^s(\alpha_1(.), \alpha_2(.))$. Bidder i's equilibrium bid $\beta_i(v)$ belongs[14] to the interior of the subinterval[15] $[\beta_i(\varphi^-(v)), r^{-1}(v)]$ of $[\beta_i(\varphi^-(v)), \beta_i(\varphi^+(v))]$.

From Corollary 1 (i) below, all bids, and not only $\beta_i(v)$, in $[\beta_i(\varphi^-(v)), r^{-1}(v)]$ are optimal for bidder i, that is, bidder i's expected payoff is constant in this interval. The reason is simple: after winning the auction with his equilibrium bid b in the interior of this interval, the more aggressive bidder j with value $\alpha_j(b)$ demands $r(b)$ as the resale price, which bidder i accepts since $r(b) < v$. Consequently, the first-order effect of a bid change by bidder i around b on his expected payoff does not depend on his value v, and hence must vanish, since b is optimal for some value.

In Example 1 all bids in $[c, r^{-1}(v_2)]$ are optimal for bidder 2. In Example 2, all bids in $[\beta_1(z), r^{-1}(v_1)]$ are optimal for bidder 1 with value $v_1 > z$.

In the SPA, from the FOC (11), $r(b) = b$ and the interval of optimal bids is $[\varphi^-(v), v]$: bidder i is indifferent between winning and losing against an opponent who submits a bid b in this interval because he ends up with the item and pays the same price b in both cases.

FIGURES 1, 2

As stated in Corollary 1 (ii), the equilibrium resale price characterizes the post-resale equilibrium allocation. If bidder 1's value is v_1, bidder 2 ends up with the item if and only if his value is above the cut-off $\lambda^s_\varphi(v_1)$ below, which is the resale price bidder 1 requires, when bidder 1 is more aggressive at v_1, and the value at which bidder 2 requires v_1, when bidder 1 is less aggressive at v_1 (see Figures 1 and 2):

[14] $\beta_i(\varphi^-(v)) < \beta_i(v)$ follows from $\varphi^-(v) < v$ and $\beta_i(v) < r^{-1}(v)$ from $\rho^s_i(v, \alpha_j(\beta_i(v))) < v$, which in turns follows from $\beta_i(v) < \beta_j(v)$.

[15] Because $r(\beta_i(\varphi^+(v))) = \varphi^+(v) > v$.

$$\begin{aligned}\lambda^s_\varphi(v_1) &= \rho^s(v_1,\varphi(v_1)), \text{ if } \varphi(v_1) \geq v_1; \\ &= \rho^s\left(\varphi^{-1}(.),.\right)^{-1}(v_1), \text{ if } \varphi(v_1) \leq v_1.\end{aligned}$$

Corollary 1: *For any PBE as in Theorem 1:*

(i) All bids in $[\beta_i(\varphi^-(v)), r^{-1}(v)]$ *are optimal for bidder* i *with value* v *in* $[c,d]$ *if he is less aggressive at* v.

(ii) If the bidders' values v_1, v_2 *are such that* $v_2 < (>) \lambda^s_\varphi(v_1)$, *then the item eventually goes to bidder 1 (2).*

Proof: See Appendix 1.

3. The FD Regime

3.1 The Ratchet Effect and Theorem 2

Bidders get no payoff from resale in any symmetric separating PBE of the SPA, since such a PBE is efficient, and a standard argument shows that the common bidding function can then not be different from the identity function. Haile (1999) already proved that the truth-bidding equilibrium of the SPA without resale is also a PBE and hence the unique symmetric separating PBE of the SPA with resale.

When resale occurs in a separating PBE, the auction winner demands as a resale price the loser's value, which he infers from the losing bid under FD. Because of the implied ratchet effect–the asymmetrical consequences of upwards and downwards deviations–, there exists no separating PBE of the SPA or FPA where bidders follow different strictly increasing and differentiable bidding functions. In fact, if small upwards deviations, which bring him no payoff at resale (he would refuse any resale offer), were not profitable to the (locally) less aggressive bidder, downwards deviations would be strictly profitable, since they would increase his payoff at resale when the auction allocation is inefficient: a nonpositive right-hand derivative of the payoff would imply a strictly negative left-hand derivative. In particular, since the same bidding function could not satisfy both bidders' (different) FOC's, there exists no pure separating (symmetric or asymmetric) PBE of the FPA with strictly increasing and differentiable bidding functions when $F_1 \neq F_2$.

Nevertheless, as we now show, there exist semi-separating behavioral PBE's where the less aggressive bidder does not completely reveal his value through his bid. A behavioral bidding strategy of bidder i is a (regular[16]) conditional probability measure $G_i(.|.)$ with respect to v_i in $[c, d]$. A resale strategy is characterized by a (measurable) function $\delta_i(.;.)$. If bidder i with value v_i follows $(G_i(.|.), \delta_i)$, he chooses his bid according to $G_i(.|v_i)$ and, after winning the auction where bidder j bids b_j, demands $\delta_i(v_i; b_j)$ at resale. Here, the second argument of the resale function δ_i of bidder i is not, as in Section 2, his own bid, but rather bidder j's bid, which he observes. We again assume that the auction loser refuses the resale price if and only if it is strictly larger than his value.

We specify the revised beliefs bidder i holds about bidder j's value after he observes bidder j's bid b_j through a (regular) conditional probability measure $F_j(.|b_j)$. We use the same notation for a probability measure and its cumulative function and we call a couple of strategies and beliefs $(G_1(.|.), \delta_1, F_2(.|.); G_2(.|.), \delta_2, F_1(.|.))$ a PBE if it can be completed into one. The randomization procedure, mentioned in Theorem 2 below, is an explicit procedure of construction of PBE's under FD that we present in the next subsection.

Theorem 2: *Let $\mathcal{E}$ be a PBE of an auction under ND as in Theorem 1. Let $\mathcal{E}'$ be the output of the randomization procedure applied to $\mathcal{E}$. Then:*

(i) $\mathcal{E}'$ is a PBE of the same auction under FD.

[16] Following the standard terminology in probability theory, this means that $G_i(.|v_i)$ is a probability measure, for all v_i, and $G_i(b|.)$ is measurable, for all b.

(ii) The bid marginal distributions, the interim total expected payoffs, and the post-resale allocation are the same in $\mathcal{E}'$ as in $\mathcal{E}$;

(iii) Conditionally on the value of the auction winner, strictly profitable resale takes place with the same probability in $\mathcal{E}'$ as in $\mathcal{E}$ and, when this probability is different from zero, at the same price;

(iv) If the auction is the SPA, the auction outcomes–the bids and the allocation before resale–are posterior implemented by $\mathcal{E}'$ or, for short, $\mathcal{E}'$ is posterior implementable.

(v) There exists a PBE of the auction under ND[17] *with the same bidding strategies and, along the equilibrium path, the same take-it-or-leave-it offers from the auction winner as in $\mathcal{E}'$.*

Following Green and Laffont (1987) (see, also, Lopomo 2001), (iv) means that all bids in the support of bidder i's bidding strategy conditional on v_i are optimal for bidder i with value v_i even after he learns bidder j's bid.

3.2 The Randomization Procedure

Let $\mathcal{E} = (\beta_1, \gamma_1; \beta_2, \gamma_2)$ be a PBE as in Theorem 1 and $\varphi = \alpha_2\beta_1$. The main idea of the randomization procedure is to have any bidder who bids less aggressively in $\mathcal{E}$ mix over his set of optimal bids (Corollary 1 (i) describes) in such a way that when the other bidder wins and observes the losing bid, he chooses the same resale price as in $\mathcal{E}$. This requirement determines the auction winner's beliefs conditional on the losing bid. The marginal bid distribution is taken to be the same as in $\mathcal{E}$. These conditional and marginal distributions then determine a joint distribution, which, using Milgrom and Weber (1985)'s terminology, is a "distributional strategy," that is, such that its marginal value distribution is the bidder's actual value distribution.

[17] (v) actually holds true for any disclosure policy, including, for example, release of "garbled" information about the bids. When $\mathcal{E}$ is asymmetric, the bidding strategies in $\mathcal{E}$ and $\mathcal{E}'$ differ.

Revised Beliefs:

C1 If $\alpha_j(b_j) \leq \alpha_i(b_j)$, $F_j(.|b_j)$ is concentrated at $\alpha_j(b_j)$.

C2 If $\alpha_j(b_j) > \alpha_i(b_j)$, the support of $F_j(.|b_j)$ is $[r(b_j), \varphi^+(\alpha_j(b_j))]$, where:

$$F_j(v_j|b_j) = 1 - \exp\int_{r(b_j)}^{v_j} \frac{dw}{\alpha_i r^{-1}(w) - w}.$$

Bidding Strategies:

B1 If $\beta_i(v_j) \leq \beta_j(v_j)$, $G_j(.|v_j)$ is concentrated at $\beta_j(v_j)$.

B2 If $\beta_j(v_j) < \beta_i(v_j)$, the support of $G_j(.|v_j)$ is $\left[\beta_j(\varphi^-(v_j)), r^{-1}(v_j)\right]$, where:

$$G_j(b|v_j) = 1 - \frac{\int_b^{r^{-1}(v_j)} \exp\int_{r(b')}^{v_j} \frac{dw}{\alpha_i r^{-1}(w) - w} dF_j(\alpha_j(b'))}{f_j(v_j)(v_j - \alpha_i r^{-1}(v_j))}. \quad (13)$$

Resale Strategies:

RS1 If $\alpha_j(b_j) \leq \alpha_i(b_j)$,

$$\begin{aligned} & \delta_i(v_i; b_j) \\ = \; & \max(r(\beta_i(v_i)), \alpha_j(b_j)), \text{ if } \beta_j(v_i) \leq \beta_i(v_i) \\ = \; & \max(\alpha_j(b_j), v_i), \text{ otherwise.} \end{aligned}$$

RS2 If $\alpha_j(b_j) > \alpha_i(b_j)$,

$$\begin{aligned} & \delta_i(v_i; b_j) \\ = \; & \max(r(\beta_i(v_i)), r(b_j)), \text{ if } \beta_j(v_i) \leq \beta_i(v_i); \\ = \; & \max(v_i, r(b_j)), \text{ otherwise.} \end{aligned}$$

From Lemma 2 (i) below, C1-C2 indeed define a conditional distribution. We now illustrate with Example 1 the proof of Lemma 2 (ii), according to which, if we choose $F_j\alpha_j$ as bidder j's marginal bid distribution, the implied joint distribution of values and bids has F_j at its value marginal distribution.

From C2, the support of $F_2(.|b_2)$ is $[r(b_2), d]$, for all b_2 in $(c, \beta_2(d))$, and, after computing $f_2(w|b_2) = \frac{d}{dw} F_2(w|b_2)$, we find:

$$1 = F_2(w|b_2) + f_2(w|b_2)\left(w - \alpha_1 r^{-1}(w)\right),$$

for all w in $[r(b_2), d]$. Integrating this equation with respect to b_2 according to the marginal $F_2\alpha_2(b_2)$ from c to $r^{-1}(w)$, we find that the joint distribution of values and bids has a marginal value distribution F_2^* with derivative f_2^* such that:

$$F_2\alpha_2\left(r^{-1}(w)\right) = F_2^*(w) + f_2^*(w)\left(w - \alpha_1 r^{-1}(w)\right),$$

for all w in (c, d). However, by definition of r, we have $w = \rho^s\left(\alpha_1 r^{-1}(w), \alpha_2 r^{-1}(w)\right)$ and hence:

$$F_2\alpha_2\left(r^{-1}(w)\right) = F_2(w) + f_2(w)\left(w - \alpha_1 r^{-1}(w)\right).$$

Subtracting this last equation from the previous one, we find:

$$\frac{d}{dw}\left(F_2^*(w) - F_2(w)\right) = \frac{F_2(w) - F_2^*(w)}{w - \alpha_1 r^{-1}(w)}.$$

Consequently, if $F_2^*(w) - F_2(w)$ was strictly positive at w, it would be strictly decreasing and strictly positive over $[c, w]$, which is impossible since $F_2^*(c) = F_2(c) = 0$. A strictly negative difference $F_2^*(w) - F_2(w)$ is similarly impossible. The marginal distribution F_2^* is then equal to the actual distribution F_2 everywhere.

In Example 2, one can easily check that B1-B2 give the conditionals of the distributional strategies, that is, that Lemma 2 (iii) below hold true. For example, bidder 2's conditional must make him bid $\beta_2(v_2)$ if $v_2 \geq z$ and randomize over $[c, r^{-1}(v_2)]$ if $v_2 < z$. For this latter case, differentiating, with respect to v_2, $F_2(v_2|b_2)$ from C2, integrating it over $[b, r^{-1}(v_2)]$ with respect to b_2, distributed according to $F_2\alpha_2$, and dividing it by $f_2(v_2)$ give the expression in B2[18].

Lemma 2:

(i) $F_j(.|b_j)$ *in C1-C2 is a (regular) conditional distribution . The distribution in C2. is absolutely continuous, with density function* $f_j(v_j|b_j)$ *equal to* $\frac{1}{v_j-\alpha_i r^{-1}(v_j)} \exp \int_{r(b_j)}^{v_j} \frac{dw}{\alpha_i r^{-1}(w)-w}$ *in the interior of the support. Furthermore,* $F_j(v_j|b_j)$ *is continuous with respect to* b_j*, for all* v_j *in* $(r(b_j), \varphi^+(\alpha_j(b_j)))$.

(ii) There exists one and only one distributional strategy of bidder j *with marginal* $F_j\alpha_j(b_j)$ *and a conditional that satisfies C1-C2.*

(iii) $G_j(b_j|v_j)$ *in B1-B2 is a (regular) conditional distribution of bidder* j*'s distributional strategy in (ii).*

(iv) $\delta_i(.;.)$ *in RS1-RS2 is optimal for bidder* i *with revised beliefs* $F_j(.|.,.)$.

(v) For all b_j *not larger than the maximum of the support of* $G_i(.|v_i)$*, we have:*

$$
\begin{aligned}
& \delta_i(v_i; b_j) \\
= \; & r(\beta_i(v_i)), \text{ if } \beta_j(v_i) \leq \beta_i(v_i); \\
= \; & v_i, \text{ otherwise.}
\end{aligned}
$$

Proof: See Appendix 2.

From Lemma 2 (iv) above, the resale prices in RS1-RS2 are optimal for the auction winner, given his revised beliefs. From Lemma 2 (v), if a bidder has won the auction by following his bidding strategy in B1-B2, the price he demands at resale does not depend on the loser's bid. We check Lemma 2

[18] In the example with different value upper extremities where v_1, v_2 are uniformly distributed over $[0,1]$ and $[0,d]$, $d > 1$, the equilibrium bidding strategies under ND are such that $\beta_1(v) = \beta_2(dv) = (1+d)v/4$. In an equivalent PBE under FD, bidder 2 bids according to $G_2(b|v_2) = (2b/v_2)^{2d/(d-1)}$ (over $[0, v_2/2]$) if $v_2 \leq (1+d)/2$ and $G_2(b|v_2) = (4b/(1+d))^{2d/(d-1)}$ (over $[0,(1+d)/4]$) if $v_2 \geq (1+d)/2$. Here, the randomization procedure does not uniquely determine $G_2(b|v_2)$ for $v_2 > (1+d)/2$.

(iv, v) for bidder 2 in Example 2. RS1-RS2 for bidder 2 with value $v_2 > z$ reduce to:

$$\begin{aligned} \delta_2(v_2; b_1) &= r(b_1), \text{ if } b_1 \geq \beta_2(v_2); \\ &= r(\beta_2(v_2)), \text{ for all } b_1 \leq \beta_2(v_2). \end{aligned}$$

If bidder 2 observes b_1 in $(\beta_1(z), \beta_2(v_2))$ from bidder 1, his revised beliefs $F_1(v_1|b_1) = 1 - \exp \int_{r(b_1)}^{v_1} \frac{dw}{\alpha_2 r^{-1}(w) - w}$ have support $[r(b_1), d]$ and virtual value:

$$\begin{aligned} & v_1 - \frac{1 - F_1(v_1|b_1)}{f_1(v_1|b_1)} \\ = \; & \alpha_2 r^{-1}(v_1), \end{aligned}$$

which is larger than v_2 if and only if v_1 is larger than $r(\beta_2(v_2))$, and $r(\beta_2(v_2))$ is bidder 2's unique optimal resale price. If bidder 2 observes b_1 in $[c, \beta_1(z)]$, no profitable resale is possible and $r(\beta_2(v_2))$ is one of bidder 2's optimal resale prices.

If, after deviating from his equilibrium bid, bidder 2 wins and observes b_1 in $(\beta_2(v_2), \beta_1(d))$, $r(b_1)$ is his unique optimal resale price because it is the minimum of the support of $F_1(.|b_1)$ and is optimal for bidder 2's larger value $\alpha_2(b_1)$. If bidder 2 observes a bid $b_1 > \beta_1(d)$, $r(d) = d$ is optimal for bidder 2's revised beliefs, which are concentrated at d.

3.3 Proof of Theorem 2

Bidder i's net-value function u_i^s–the difference between his utility $u_i^{s,w}$

from winning and his utility $u_i^{s,l}$ from losing–is as follows[19]:

$$\begin{aligned} & u_i^s(v_i, v_j; b_i, b_j) \\ = & \delta_i(v_i; b_j), \text{ if not larger than } v_j; \\ = & \delta_j(v_j; b_i), \text{ if not larger than } v_i; \\ = & v_i, \text{ otherwise.} \end{aligned}$$

Here, contrary to Section 2, bidder i's utility in case of winning $u_i^{s,w}$ depends on bidder j's bid b_j, which bidder i observes after winning, and is independent of bidder i's own bid b_i. Thus we may consider bidder i's winning with probability one as his status quo and his net expected utility is $\int_d^{b_i} \int u_i^s(v_i, v_j; b_i, b_j)\, dF_j(v_j|b_j)\, dF_j\alpha_j(b_j)$, where, according to the randomization procedure, bidder j's marginal bid probability distribution in $\mathcal{E}'$ is the same–$F_j\alpha_j$–as in $\mathcal{E}$.

Lemma 3 below is similar to Lemma 1 and follows from the randomization procedure. Lemma 3 (i.1) and (i.2) come from the independence, by Lemma 2 (v), of the resale price on the auction loser's bid, if the auction winner follows his bidding strategy in $\mathcal{E}'$. Lemma 3 (ii) holds true because, although the bidders use behavioral bidding strategies, resale remedies any inefficient outcome after a tie. For example, when bidder 2 with value v_2 in (z, d) in Example 2 wins after tying at $\beta_2(v_2)$, he demands the resale price $r(\beta_2(v_2))$, which bidder 1 accepts with probability one because it is the minimum of the support of $F_1(.|b_1 = \beta_2(v_2))$. The monotonicity in Lemma 3 (iii) follows from the monotonicity of a bidder's optimal expected payoff at resale with respect to his value.

Lemma 3:

[19]Since we only consider β_i, β_j from a given $\mathcal{E}$, we drop them from the argument of the net values u_i^s.

(i.1) For all (v_i, b_i) *in* $[c,d] \times [c,+\infty)$ *and all* $b_j \geq b_i$*:*

$$\int u_i^s (v_i, v_j; b_i, b_j)\, dF_j (v_j|b_j) = \int u_i^s (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j)$$

(i.2) For all (v_i, b_i) *in* $[c,d] \times [c,+\infty)$*:*

$$\int_d^{b_i} \int u_i^s (v_i, v_j; b_i, b_j)\, dF_j (v_j|b_j)\, dF_j\alpha_j (b_j) = \int_d^{b_i} \int u_i^s (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j)\, dF_j\alpha_j (b_j).$$

(ii) For all $b \geq c$*:*

$$\int u_i^s (\alpha_i (b), v_j; b, b)\, dF_j (v_j|b) = r(b).$$

(iii) For all $b \geq c$*,* $\int u_i^s (v_i, v_j; b, b)\, dF_j (v_j|b)$ *is nondecreasing with respect to* v_i *in* $[c,d]$ *and continuous with respect to* b*.*

Proof: See Appendix 2.

The proof in Appendix 2 of Theorem 2 (i) is similar to the proof, in Section 2, of Theorem 1. Consider bidder 1 in Example 1 with value v_1 in (c,d). His expected payoff in the FPA is quasi-concave with respect to his bid with a maximum at $b = \beta_1(v_1)$. In fact, its derivative is nonnegative below $\beta_1(v_1)$ and nonpositive above it. For example, the derivative at b in $(c, r^{-1}(v_1))$ is, up to the factor $F_2\alpha_2(b)$:

$$\left\{ \begin{array}{c} \int_{r(b)}^{v_1} v_2 dF_2 (v_2|b) \\ +v_2 (F_2 (r(\beta_1(v_1)) |b) - F_2 (v_1|b)) \\ +r(\beta_1(v_1)) (1 - F_2 (r(\beta_1(v_1)) |b)) \\ -b \end{array} \right\} \frac{d}{db} \ln F_2\alpha_2 (b)$$
$$-1.$$

The first three terms in the factor between braces above is the net expected utility at b: the first term is the expected resale price bidder 1 saves by winning at auction rather than buying at resale (bidder 2 demands his value as the resale price), the second and third terms are the expected payoff bidder 1 receives from reselling the item (bidder 1 demands $r(\beta_1(v_1))$ after winning the auction). Since this net expected utility is nondecreasing in v_1 (see the general proof in Appendix 2), it is not smaller than $r(b)$, what it would be at $\alpha_1(b) < v_1$. The derivative above is then not smaller than $(r(b) - b) \frac{d}{db} \ln F_2\alpha_2(b) - 1$, which is equal to zero.

Contrary to Lemma 1 (i), the similar property for the ND regime, Lemma 3 (i.2) does not follow from an envelope theorem, but rather from the identity (i.1): $\int u_i^s(v_i, v_j; b_j, b_j)\, dF_j(v_j|b_j)$ is equal to the actual net utility bidder i receives from submitting $b_i \leq b_j$ when his opponent submits b_j. The quasi-concavity (which follows from Lemma 3) of bidder i's expected payoff[20] in the SPA implies the no-regret property: bidder i wins against all those bids b_j that contribute nonnegatively to his payoff and loses against those that would contribute nonpositively.

Proof of Theorem 2: See Appendix 2.

Although the final allocations in $\mathcal{E}'$ and $\mathcal{E}$ are identical, the intermediate allocations, after the auction and before resale, differ with strictly positive probability if the equilibria are asymmetric. For example, in Figure 1, if if bidder 1's value is v_1 and bidder 2's value is in $\left(\lambda_\varphi^s(v_1), \varphi(v_1)\right)$, bidder 1 wins the auction in $\mathcal{E}$ with probability one and loses it with strictly positive probability in $\mathcal{E}'$.

3.4 Multiplicity of PBE's of the SPA

Beyond the multiplicity in Theorem 2, we construct in Appendix 3 further PBE's of the SPA by extending Theorem 1 and the randomization procedure to nondecreasing, and possibly discontinuous, functions φ. For example, in

[20] Which is, up to a term constant in b_i, $\int^{b_i} \left(\int u_i^s(v_i, v_j; b_j, b_j)\, dF_j(v_j|b_j) - b_j\right) dF_j\alpha_j(b_j)$.

the PBE constructed from the function φ such that $\varphi(v_1) = \theta^*$, for all v_1 in $[c, \theta^*]$, and $\varphi(v_1) = v_1$, for all v_1 in $[\theta^*, d]$: the bidding functions are such that $\beta_1(v_1) = \rho^s(v_1, \theta^*)$, for all v_1 in $[c, \theta^*]$, and $\beta_1(v_1) = v_1$, for all v_1 in $[\theta^*, d]$; $\beta_2(v_2) = \rho^s(c, v_2)$, for all v_2 in $[c, \theta^*)$, $\beta_2(\theta^*)$ belongs to $[\rho^s(c, \theta^*), \theta^*]$, and $\beta_2(v_2) = v_2$, for all v_1 in $(\theta^*, d]$. While our randomization procedure produces an equivalent behavioral PBE where bidder 2 with value v_2 in $[c, \theta^*)$ randomizes over $[c, v_2]$, we can make him bid[21] c instead and obtain Garratt *et al* (2006b)'s pure PBE under FD and no discounting[22]. If $\theta^* = d$, we obtain the "extreme" PBE where a bidding function takes the constant value d and the other takes the constant value c[23]. From Garratt *et al* (2008), some lotteries over such PBE's, for example, the lottery that gives probability 1/2 to each extreme PBE when the values are identically and uniformly distributed, dominate the truth-bidding PBE with respect to the bidders' interim payoffs[24].

Hafalir and Krishna (2008) prove that the truth-bidding PBE is the only PBE where no bidder would regret his resale offer, nor his bid, upon learning the other bidder's value. Of course, the values being private, a bidder may actually never learn the other bidder's value. From Theorem 2 (iv), all our PBE's satisfy the weaker no-regret property according to which no bidder will regret his bid after he learns the other bidder's bid, which he does if he wins the auction. Moreover, any of our PBE's remains a PBE after a change

[21] Since c is among his optimal bids, bidder 2 has no incentive to deviate. Bidder 1 has no incentive to deviate from his bidding strategy because by submitting a bid in $(c, \rho^s(c, \theta^*))$, where bidder 2's bid distribution has changed, he now obtains the same expected payoff he previously obtained by submitting a bid in $(\rho^s(c, \theta^*), \theta^*)$. Deviating to c is also unprofitable, since bidding slightly above it is at least as advantageous (depending on the tie-breaking rule).

[22] A similar construction when the auction loser sets the resale price (see Section 4) gives the bidding functions $\beta_1(v_1) = v_1$, for all v_1 in $[c, \theta^*)$; $\beta_1(v_1) = d$, for all v_1 in $(\theta^*, d]$; $\beta_2(v_2) = v_2$, for all v_2 in $[c, \theta^*)$; $\beta_2(v_2) = \rho^b(\theta^*, v_2)$, for all v_2 in $(\theta^*, d]$.

[23] This is a PBE for any (voluntary) bargaining at resale.

[24] Bidders might then want to implement such a combination of PBE's, if they have access to a common randomizing device.

of the value distribution of the more aggressive bidder[25].

As in the common-value model (Bikchandani 1988, Klemperer 1998, Bulow *et al* 1999), the multiplicity of PBE's of the SPA reflects the sensitivity of the outcome to small changes in the rules or payoffs. For example, as in k-price auctions, inject an element of FPA by having bidder i, if he wins, pay the weighted average $k_i b_i + (1 - k_i)\, b_j$ of his bid and the second highest bid, where k_i is small. We show in Section 6 that the then only differentiable PBE under ND is payoff equivalent to the PBE of the unaltered SPA that is constructed from $\varphi = F_2^{-1} F_1^l$, where $l = k_1/k_2$. Furthermore, this particular PBE of the SPA is the limit of the PBE of the hybrid auction, as k_1 and k_2 tend towards zero while the ratio k_1/k_2 stays constant at l[26]. The randomization procedure would also produce PBE's of the hybrid auctions under FD.

See Section 7 for selection criteria from the literature on auctions with common value.

3.5 No Other PBE's Within a Restricted Class

We now show that the randomization procedure produces all the PBE's under FD that satisfy the following assumptions, for all i, j, $i \neq j$:

A1 Bidder i's marginal bid distribution has its support equal to an interval $\left[c, \overline{d}\right]$, with $\overline{d} > c$, and its cumulative function $G_i\left(b_i\right)$ is continuously differentiable over $\left(c, \overline{d}\right]$.

A2 When mutually strictly profitable resale is possible according to $F_j\left(.|b\right)$, $\delta_i\left(v, b\right)$ is bidder i's unique optimal resale price.

A3 The support of bidder i's distributional strategy is the closed set bordered by the graphs of two functions β_i^u and β_i^l, with $\beta_i^u \geq \beta_i^l$.

[25] See Corollary A3.1 in Appendix 3.

[26] Similar small perturbations in the rules might be used by an auctioneer who would want to implement a particular PBE (for example, for revenue purposes).

A4 β_i^u and β_i^l are nondecreasing.

A5 $\beta_1^u(c) = \beta_2^u(c) = \beta_1^l(c) = \beta_2^l(c) = c$ and $\beta_1^u(d) = \beta_2^u(d) = \overline{d}$.

A6 β_1^u, β_2^u are differentiable over $(c, d]$ with strictly positive derivatives.

A7 For all open subinterval (v', v'') of $[c, d]$ such that $\beta_i^u(v) \geq \beta_j^u(v)$, for all v in (v', v''):

A7.0 The support of $G_j(.|v)$ is $\left[\beta_j^l(v), \beta_j^u(v)\right]$

A7.1 $\beta_i^u(v) = \beta_i^l(v)$; and if $\beta_i^u > \beta_j^u$ everywhere over (v', v''), then $\beta_j^u > \beta_j^l$ everywhere over the same interval;

A7.2 β_l^l is differentiable over (v', v'');

and, for all (v_i, b) in $(v', v'') \times (\beta_i^u(v'), \beta_i^u(v''))$:

A7.3 $\delta_i(v; b)$ is continuous with respect to b at (v_i, b) if $b \leq \beta_i^u(v_i)$;

A7.4 $\delta_i(v; b)$ is continuous with respect to v and differentiable with respect to b at (v_i, b) if $b \leq \beta_i^u(v_i)$ and (v_i, b) belongs to the interior of the support of bidder j's distributional strategy.

Because of the rachet effect, there can only exist PBE's where the less aggressive bidder follows a behavioral strategy. Moreover if a bidder follows a pure bidding strategy, the graph of his bidding function cannot intersect the interior of his opponent's distributional strategy. If it was the case, he would be a buyer at resale with a strictly positive probability and the ratchet effect would make impossible to prevent deviations. Hence, we assume, in A7.1, that, when the bidders do not use the same bidding function, the less aggressive bidder randomizes over a range of bids and the more aggressive bidder follows a bidding function above this range[27].

A3, A5, and A7.0 are assumptions of "full support." A4 is a monotonicity assumption. The uniqueness of the optimal resale price, when profitable resale is possible, also implies its monotonicity with respect to the price setter's value. A1, A5, A6, and A7.2 to A7.4 are continuity and differentiability assumptions. We have the remark and Theorem 3 below.

[27]We could have shortened A7.1 to only $\beta_i^u(v) = \beta_i^l(v)$. The second part of A7.1 would then have followed from the ratchet effect.

Remark: A3, A4, A5, and A7.1 imply that β_i^l is continuous at all $v < d$ such that $\beta_i^u(v) = \beta_j^u(v)$.

Theorem 3: *Let $\mathcal{E}'$ be a PBE of an auction under FD that satisfies Assumptions A1 to A7. Then, there exists a PBE $\mathcal{E}$ as in Theorem 1 of the auction under ND such that $\mathcal{E}$ and the PBE under FD produced by the randomization procedure applied to $\mathcal{E}$ are equivalent, in the sense of (i) and (ii) in Theorem 2, to $\mathcal{E}'$.*

The main tools of our proof are the ratchet effect and the following "probability invariance," which is a direct consequence of the envelope theorem. It states that, in the interior of the support of bidder i's distributional strategy, the probability $\Pr_i(w, b)$ that he becomes the eventual owner of the item is independent of his bid b.

Proposition: (Probability Invariance) *If $(w, w') \times (b, b')$ is included in the interior of the support of bidder i's distributional strategy in a PBE, then* $\Pr_i(w'', b'') = \Pr_i\left(w'', \widetilde{b}\right)$*, for all $b'', \widetilde{b}$ in (b, b') and almost-all w'' in (w, w').*

Proof:[28] From the assumptions above, within the interior of the support of his distributional strategy, the less aggressive bidder's expected payoff is a continuous function of his bid[29]. From A7.0, all bids in (b, b') must give the same optimal payoff $P_i(w'')$ to bidder i with value w'' in (w, w').

Let v_i, v_i' be in (w, w'). Since, for all (w'', b'') in $(w, w') \times (b, b')$, bidder i with value w'' obtains $P_i(w'')$ from submitting b'', we have, from Myerson (1981):

$$P_i(v_i') - P_i(v_i) = \int_{v_i}^{v_i'} \Pr_i(w'', b'')\, dw'',$$

[28] In the proof, we only make use of the continuity of a bidder's expected payoff with respect to his bid.

[29] The expected payoff from the auction stage is obviously continuous. The expected payoff from the resale stage $\int \int I(b_j > b; v' < v_j < v'') \max(v_i - \delta_j(v_j; b), 0)\, dF_j(v_j|b_j)\, dG_j(b_j)$ is also continuous since the function inside the integral is continuous at b, for all v_j (from A7.3) and almost all b_j, and is bounded.

and, consequently:

$$\int_{v_i}^{v_i'} \left(\Pr_i (w'', b'') - \Pr_i \left(w'', \widetilde{b} \right) \right) dw'' = 0,$$

for all $b'', \widetilde{b}$ in (b, b') and v_i, v_i' in (w, w'), which implies $\Pr_i (w'', b'') = \Pr_i \left(w'', \widetilde{b} \right)$, for almost-all w'' in (w, w'). ||

We sketch below the remaining main steps of the proof, whose details can be found in Appendix 4.

Step 1: *After winning a tie at auction, the more aggressive bidder demands a resale price that is accepted with probability one.* (Lemma A4.1)

Sketch of the proof: Otherwise, there would be values for which the less aggressive bidder randomizes that are smaller than the resale price. Since he would refuse resale offers, this bidder could then obtain the item only by winning the auction, the probability of which is not constant (since it decreases with his bid), and it would contradict probability invariance (See Figure A1).

Step 2: *If the couple formed by the resale price of the more aggressive bidder and the less aggressive bidder's bid belongs to the interior of the support of this latter bidder's distributional strategy, the resale price is independent of the bid* (Lemma A4.2).

Sketch of the proof: Otherwise, the graph of the resale price, as function of the less aggressive bidder's bid, would not be vertical and some changes of bids, within the support of the distributional strategy, would cross these graphs and result in different probabilities of the less aggressive bidder's getting the item (see Figure A2).

From Steps 1 and 2, as the bid from the less aggressive bidder decreases starting from a tie, the graph of the resale price starts from the upper boundary of the support of his distributional strategy, may follow this upper boundary, is vertical after it enters the interior of the support, and may then follow the lower boundary (Lemma A4.3).

Step 3: *For a fixed value of the more aggressive bidder, the graph of the resale price he demands does not follow the upper boundary of the support of the less aggressive bidder's strategy* (Lemma A4.4).

Sketch of the proof: Otherwise, a "modified" ratchet effect would exist: although deviations from the same bid starting from couples in the interior and on the upper boundary of the bid support would have the same effect–proportional to the difference between the resale price (which the less aggressive bidder accepts, from Step 1) and the bid–through the change of auction outcome, a deviation from the interior has an effect through the resale price when the less aggressive bidder keeps losing, while an upward deviation from the upper boundary does not (since the less aggressive bidder refuses any resale price) (See Figure A3).

Step 4: *The lower boundary of the less aggressive bidder's support is flat* (Lemma A4.5).

Sketch of the proof: Otherwise, a ratchet effect would exist: a downward bid deviation from a point on the lower boundary would change the less aggressive bidder's payoff from resale, while an upward deviation would not (See Figure A4).

From Steps 3 and 4, if bidder i is more aggressive at v_i, the resale price bidder i demands is equal to $\alpha_j^u \beta_i^u (v_i)$, where α_j^u is the inverse of β_j^u, and is independent of his opponent's bid when profitable resale is possible. Since $\alpha_j^u (\beta_i (v_i)) \geq v_i$, we then have:

$$\alpha_j^u (\beta_i (v_i)) \in \arg\max_p (p - v_i) (1 - F_j (p|b)),$$

for all $b \leq \beta_i (v_i)$. Integrating the objective function over this range of bids with respect to bidder j's marginal bid distribution G_j, we find:

$$\alpha_j^u(\beta_i(v_i)) \in \arg\max_p (p - v_i)(G_j(\beta_i(v_i)) - F_j(p)). \quad (14)$$

The final step of the proof is Step 5 below.

Step 5: ("Inverse randomization procedure") *The PBE is equivalent to the PBE under ND with bidding functions* β_1^*, β_2^* *where* $\beta_i^* = G_i^{-1} F_i$ *(thus* $\beta_i^*(v_i) = \beta_i(v_i)$*, if bidder* i *is more aggressive at* v_i*), where* G_i *is bidder* i*'s marginal bid distribution* (Lemmas A4.6 and A4.7).

Sketch of the proof: From (14) above and $G_j(\beta_i(v_i)) = F_j(\alpha_j^*(\beta_i(v_i)))$, we have $\alpha_j^u(\beta_i(v_i)) = \rho_i^s(v_i, \alpha_j^* \beta_i(v_i))$. Because resale always occurs after an inefficient resolution of a tie, the functions β_1^*, β_2^* then satisfy the same FOC's as the PBE's under ND.

From Theorem 2 and Subsection 3.4, there exist PBE's of the SPA that are not differentiable or even exhibit discontinuities, and nonconvex bid supports, and hence that do not belong to the restricted class of this subsection. We do not address the existence of PBE's of the FPA outside this class.

3.6 Revenue Comparisons Across Auctions

Since none of our PBE's gives a positive payoff to any bidder with the smallest value c, the auctioneer's expected revenues are, from Myerson (1981), the expectation of the eventual owner's virtual value. Without loss of generality, assume that bidder 1 has the higher virtual value when both bidders' values are c or, equivalently, $f_1(c) \geq f_2(c)$. Let ψ be the function that links the bidders' values with the same virtual value, that is, such that:

$$\psi(v) - \frac{1 - F_2(\psi(v))}{f_2(\psi(v))} = v - \frac{1 - F_1(v)}{f_1(v)}, \quad (15)$$

for all v in $[c, d]$. From Corollary 1 (ii), a PBE's final allocation is characterized by the function λ_φ^s, with $\varphi = \alpha_2 \beta_1$. If λ_φ^s was equal to ψ, the PBE would maximize revenues. Under the assumption of differentiability of ψ, it is simple to prove (Lemma A6.1 in Appendix 6) that, when the bidders are heterogeneous, this is not the case for the FPA, where, from Theorem 1, $\varphi = F_2^{-1}F_1$. Thus, there exists an interval where λ_φ^s, with $\varphi = F_2^{-1}F_1$, is everywhere different from ψ. By slightly moving φ over this interval, while keeping it continuous and strictly increasing, towards and away from ψ, one makes λ_φ^s move in the same direction and Corollary 2 follows.

Corollary 2: *When ψ is differentiable and $F_1 \neq F_2$, the revenues from the PBE of the FPA are strictly smaller than the revenues from some PBE's of the SPA and strictly larger than the revenues from some others*[30].

With homogeneous bidders, the PBE of the FPA and the truth-bidding PBE of the SPA are efficient and maximize revenues.

4. The Other Bargaining Procedure at Resale

When the seller-virtual-value functions $v + \frac{F_1(v)}{f_1(v)}$ and $v + \frac{F_2(v)}{f_2(v)}$ are strictly increasing, we can apply the methods above to auctions after which the auction loser sets the resale price by using, instead of ρ^s, the buyer's optimal-resale-price function[31] ρ^b, whose value at (w_1, w_2), is the unique solution to the equation below:

$$w_l = \rho^b(w_1, w_2) - \frac{F_k(w_k) - F_k(\rho^b(w_1, w_2))}{f_k(\rho^b(w_1, w_2))},$$

with k and l such that $\{k, l\} = \{1, 2\}$ and $w_k \leq w_l$. Results similar to all the results above (including Corollary 2) hold true. The final equilibrium

[30] Obviously, an extreme PBE of the SPA as in Subsection 3.4 gives lower revenues–c– than the PBE of the FPA.

[31] (2) becomes $\gamma_i(v; b) = \rho_i^b(v, \min(v, \alpha_j(b)))$.

allocation is characterized by λ_{φ}^{b} (instead of λ_{φ}^{s}) such that::

$$\begin{aligned}\lambda_{\varphi}^{b}(v_1) &= \rho^{b}(v_1, \varphi(v_1)), \text{ if } \varphi(v_1) \leq v_1; \\ &= \rho^{b}\left(\varphi^{-1}(.), .\right)^{-1}(v_1), \text{ if } \varphi(v_1) \geq v_1.\end{aligned}$$

The randomization procedure makes the more aggressive bidder, who is here the equilibrium resale-price taker, randomize over his set of optimal bids– $\left[\left(r^{b}\right)^{-1}(v), \beta_i(\varphi^{+}(v))\right]$, where $r^{b} = \rho^{b}(\alpha_1, \alpha_2)$, if his value is v. In the definition of the restricted class in Subsection 3.5, within which we characterize all PBE's, Assumptions A6^{b}, A7^{b} below should replace A6, A7.

A6^{b} β_1^{l}, β_2^{l} are differentiable over $(c, d]$ with strictly positive derivatives.

A7^{b} For all open subinterval (v', v'') of $[c, d]$ such that $\beta_i^{l}(v) \geq \beta_j^{l}(v)$, for all v in (v', v''):

A7.0 The support of $G_i(.|v)$ is $\left[\beta_i^{l}(v), \beta_i^{u}(v)\right]$

A7.1 $\beta_j^{u}(v) = \beta_j^{l}(v)$; and if $\beta_i^{l} > \beta_j^{l}$ everywhere over (v', v''), then $\beta_i^{u} > \beta_i^{l}$ everywhere over the same interval;

A7.2 β_i^{u} is differentiable over (v', v'');

and, for all (v_j, b) in $(v', v'') \times \left(\beta_j^{l}(v'), \beta_j^{l}(v'')\right)$:

A7.3 δ_j is continuous with respect to b at (v_j, b) if $b \geq \beta_j^{l}(v_j)$;

A7.4 δ_j is continuous with respect to v and differentiable with respect to b at (v_j, b) if $b \geq \beta_j^{l}(v_i)$ and (v_j, b) belongs to the interior of the support of bidder i's distributional strategy.

5. Revenue Comparisons Across Bargaining Procedures

In this section, we assume that $f_1(c) \geq f_2(c)$ and that both seller-virtual-value-functions and buyer-virtual-value-functions are strictly increasing. When the bidders are homogeneous, the PBE of the FPA under both bargaining procedures reduces to the equilibrium of the FPA with no resale allowed, and we have Corollary 3 (i) below. Although, as simple examples show (see Appendix 5), the equilibrium bid distributions do not generally dominate those when the auction loser sets the resale price, we prove below that the expected revenues are strictly higher when the auction winner sets the resale price following the FPA and the bidders are heterogenous[32].

Corollary 3:

(i) When $F_1 = F_2$, the revenues from the FPA are the same whether the auction winner or the auction loser sets the resale price.

(ii) When $F_1 \neq F_2$, the revenues from the PBE of the FPA are strictly higher when the auction winner sets the resale price.

Proof: **(ii)** The PBE of the FPA allocates the item according to the function $\lambda^s_{F_2^{-1}F_1}$, if the auction winner sets the resale price, and $\lambda^b_{F_2^{-1}F_1}$, if the auction loser sets it. According to Lemma 4 below, wherever the final allocations differ, the PBE under the former bargaining procedure chooses the bidder with higher (buyer-) virtual value as the eventual owner of the item. Moreover, combined with Lemma A6.1 in Appendix 6, according to which $\lambda^s_{F_2^{-1}F_1} \neq \psi$, it also implies that the allocations do differ. (ii) then follows from Myerson (1981). ||

Lemma 4: *For all v in $[c, d]$, if $F_1(v) = F_2(v)$ then $\lambda^b_{F_2^{-1}F_1}(v) = \lambda^s_{F_2^{-1}F_1}(v)$ and if $F_1(v) \neq F_2(v)$:*

$$\lambda^b_{F_2^{-1}F_1}(v) < (>) \lambda^s_{F_2^{-1}F_1}(v) \text{ if and only if } \lambda^s_{F_2^{-1}F_1}(v) < (>) \psi(v).$$

Proof: See Appendix 6.

[32] In the uniform example of Footnote 18, we have, under ND: $\beta_1(v) = \beta_2(dv) = (1+d)v/4$, if $v \leq 2/(1+d)$; $1 - 1/(v(1+d))$, if $v \geq 2/(1+d)$ (in the formula similar to (3), the optimal resale price $\rho^b\left(F_1^{-1}(q), F_2^{-1}(q)\right)$ is the corner solution 1 when $q > 2/(1+d)$). The randomization procedure makes bidder 1 bid, under FD, over $[v/2, d/(d+1)]$ according to: $G_1(b|v_1) = 1 - (v/2b)^{2/(d-1)}$, if $v/2 \leq b \leq 1/2$; $1 - v^{2/(d-1)}\{d+1-1/(1-b)\}/(d-1)$, if $1/2 \leq b \leq d/(d+1)$.

We now present assumptions under which the equilibrium revenues from the SPA have the same range under both bargaining procedures. Assume the auction winner sets the resale price. If there exists a nondecreasing function φ^* such that $\lambda^s_{\varphi^*} = \psi$, then the PBE of the SPA constructed from a strictly increasing and continuous function φ, with $\varphi(c) = c$ and $\varphi(d) = d$, that is close to φ^* gives revenues close to the optimal revenues since its final allocation λ^s_φ is close to ψ. Assumption A^s below, which extends Zheng (2002)'s "Resale Monotonicity Assumption" to unranked hazard rates in the two-bidder case, is adapted from Lebrun (2008) and guarantees the existence of such a function φ^*. According to Assumption A^s, when the optimal allocation is biased in favor of bidder 1, for example, for some value v, there exists an intermediate allocation that is further biased in favor of this bidder and is defined by a nondecreasing function such that, after receiving the item at the intermediate stage, he offers $\psi(v)$ as the resale-price.

Assumption A^s:

(i) The unique continuous function μ^s_2 defined over $C = \{v \in [c,d] \,|\, \psi(v) \geq v\}$ and such that $\mu^s_2(v) \geq \psi(v)$ and $\rho^s(v, \mu^s_2(v)) = \psi(v)$, for all v in C, is nondecreasing.

(ii) The unique continuous function μ^s_1 defined over $D = \{v \in [c,d] \,|\, \psi(v) \leq v\}$ and such that $\mu^s_1(v) \geq \psi^{-1}(v)$ and $\rho^s(\mu^s_1(v), v) = \psi^{-1}(v)$, for all v in D, is nondecreasing.

The existence of the functions μ^s_1, μ^s_2 as defined above comes from the continuity and strict monotonicity of ρ^s. The function φ^* can be constructed as follows: $\varphi^* = \mu^s_2$ over C and $\varphi^* = (\mu^s_1)^{-1}$ over $\mu_1(D)$. Corollary 4 below follows.

Corollary 4: *Let the auction winner set the resale price and let Assumption A^s be satisfied. Then, for any incentive-compatible and individually rational mechanism that sells the item with probability one, there exist PBE's of the SPA that give either higher or arbitrarily close revenues.*

Assumption A^b, again adapted from Lebrun (2008), and Corollary 5 below are relevant to the other bargaining procedure at resale[33].

Assumption A^b: Let $f_1(c)$ and $f_2(c)$ be equal.

(i) The unique continuous function μ_1^b defined over C and such that $\mu_1^b(v) \leq \psi^{-1}(v)$ and $\rho^b\left(\mu_1^b(v), v\right) = \psi^{-1}(v)$, for all v in C, is nondecreasing.

(ii) The unique continuous function μ_2^b defined over D and such that $\mu_2^b(v) \leq \psi(v)$ and $\rho^b\left(v, \mu_2^b(v)\right) = \psi(v)$, for all v in D, is nondecreasing.

Corollary 5: *Let the auction loser set the resale price and let Assumption A^b be satisfied. Then, for any incentive-compatible and individually rational mechanism that sells the item with probability one, there exist PBE's of the SPA that give either higher or arbitrarily close revenues.*

6. (k_1, k_2)-Price Auctions.

For k in $(0,1)$, Güth and van Damme (1986) define the k-price auction, or k-PA, as the auction where the highest bidder wins and pays the weighted average of his bid and the second highest bid with respective weights k and $1-k$. If $k=0$, the k-PA is the SPA and if $k=1$ it is the FPA. We further extend the definition to allow discriminatory auction rules. In a (k_1, k_2)-price auction, or (k_1, k_2)-PA, the auction price is computed with the weights k_1 and $1-k_1$ when bidder 1 wins and k_2 and $1-k_2$ when bidder 2 wins.

The analysis of the FPA easily extends to such (k_1, k_2)-PA's, where k_1, k_2 belong to $(0,1]$. For example, under FD, the FOC becomes, where $x = s$

[33] If $f_1(c) > f_2(c)$, then $\psi(c) > c$ and there exists no φ^* such that $\lambda_{\varphi^*}^b = \psi$. In fact, if there existed such a function φ^*, one would have $c \leq (\varphi^*)^{-1}(v) = \mu_1^b(v) \leq \psi^{-1}(v)$, for all v in $E = \{v \in [\psi(c), d] \,|\, \psi(v) \geq v\}$, and thus $\varphi^*(c) = \psi(c) > c$. Consequently, bidder 2 with value $\psi(c) > c$ would choose, as a buyer at resale, the price c when he believes that bidder 1's value is distributed over $[c, d]$. Since this resale price would never be accepted, it would clearly not be optimal. In Assumption A^b, we rule out the inequality $f_1(c) > f_2(c)$ by assuming rather $f_1(c) = f_2(c)$. Note that φ^* should then be defined as $\left(\mu_1^b\right)^{-1}$ over C and μ_2^b over D.

if the auction winner sets the resale price or b if the auction loser sets the resale price:

$$\frac{d}{db}\ln F_i\left(\alpha_i\left(b\right)\right)^{1/k_j} = \frac{1}{\rho^x\left(\alpha_1\left(b\right),\alpha_2\left(b\right)\right)-b},\ i,j=1,2,\ \neq j,\ (16)$$

which[34] implies $F_1^{1/k_2}\left(\alpha_1\right) = F_2^{1/k_1}\left(\alpha_2\right)$, or, equivalently:

$$\alpha_2\beta_1\left(v\right) = F_2^{-1}\left(F_1\left(v\right)^{k_1/k_2}\right).(17)$$

The equilibrium biding function are then:

$$\beta_i\left(v\right) = \frac{\int_0^{F_i(v)^{1/k_j}}\rho^x\left(F_1^{-1}\left(q^{k_2}\right),F_2^{-1}\left(q^{k_1}\right)\right)dq}{F_i\left(v\right)^{1/k_j}},\ i,j=1,2,\ i\neq j.(18)$$

A theorem similar to Theorem 1 can be proved along the same lines. Our randomization procedure produces a PBE under FD that is equivalent to the PBE under ND.

From (17), the function φ that determines the post-auction allocation and hence the function λ_φ^x that determines the final allocation depend only on the ratio k_1/k_2. Consequently, all k-PA's, where $k_1 = k_2 = k$, give the same payoffs as the FPA. As another particular consequence, for all ς in $(0,1)$, the (k_1,k_2)-PA and the $(\varsigma k_1,\varsigma k_2)$-PA give the same payoffs to the bidders and the auctioneer as the SPA where the bidders follow the PBE that is constructed from the function $\varphi = F_2^{-1}F_1^{k_1/k_2}$. In Appendix 7, we prove that, if ζ tends towards zero, the PBE of the $(\varsigma k_1,\varsigma k_2)$-PA tends towards this payoff-equivalent PBE of the SPA.

7. N-Bidder Extensions

We extend the model to n bidders with values independently distributed over $[c,d]$ according to distributions $F_1,...,F_n$ that satisfy the same assump-

[34] With the boundary condition $\beta_1\left(d\right) = \beta_2\left(d\right)$.

tions we made in the two-bidder model. In order to avoid the problem of competing principals, we only consider the bargaining procedure where the auction winner selects some bidders to take part in his "Myerson mechanism," that is, his optimal resale mechanism among all "transparent" (whose rules can be applied independently of his private information) mechanisms that forbid resale[35].

7.1 EA

Consider the standard English auction, or EA, with irrevocable exit and full information about the bidder's activities, identities, and drop out prices[36], where the price starts rising from c. Assume that even the inactive bidders observe the entire public history of the auction.

If, as it is customary, we interpret a bidder's bid as the price at which he exits the auction if the other bidder has not yet, all our PBE's of the SPA under FD with $n = 2$ bidders are also PBE's of the EA. It is then straightforward to construct a multiplicity of PBE's with $n > 2$ bidders by having all bidders follow the standard strategy–to drop out when the price reaches one's value–until all biders but two have dropped out. At this final stage, when the price rises from the last drop out price w, the two remaining bidders follow (if they have not previously deviated) one of our PBE's when the values are distributed over the truncated interval $[w, d]$[37]. The function φ_w used to construct this PBE may depend on the drop out price w and also on the entire previous public history.

The revised beliefs about the value of a bidder who dropped out before the final stage are concentrated at his drop out price. From the randomization procedure, the virtual value of either of the two remaining bidders conditional on his bid is at least equal to w and hence to the virtual value of

[35]In footnotes, we show that the auction winner, if able to commit, can implement this mechanism even when further resale (among the auction losers) is allowed.

[36]Version (1) in Bikhchandani and Riley (1991).

[37]The (seller) virtual values remain strictly increasing after such a truncation.

any bidder who dropped out earlier. Consequently, the auction winner, even if he has previously deviated from his strategy, wants to include in his resale mechanism only his opponent in the final stage. In particular, it is optimal for any two remaining bidders who have not previously deviated to follow the 2-bidder PBE[38]. In this final stage PBE, the equilibrium and hence the optimal payoff of any bidder with value w is zero.

For a bidder's deviation in a stage prior to the last stage to have an effect on his payoff, it must either:

(i) make him active at the final stage while he is supposed to drop out earlier;

or

(ii) make him drop out while he is supposed to be active in the last stage.

In (i) his actual value is not larger than the level w of the price at the start of the final stage, following his deviation. The most he can obtain in this final stage can then not exceed his optimal payoff if his value was equal to this larger level. Since this optimal payoff is zero[39], no such deviation is profitable. In (ii), the deviation, since it brings him no payoff, cannot be profitable. Consequently, such strategies and beliefs form a PBE.

7.2 FPA

Consider the subclass of models where:

(i) $F_1 = ... = F_{n-1} = F$;

(ii) $F_n = H$ with density h that is continuously differentiable over $(c, d]$ and such that $\frac{h}{H}$ is nondecreasing and the derivative of $v - \frac{1-H(v)}{h(v)}$ is strictly positive over $(c, d]$;

[38] Since the bidders who have previously dropped out have smaller values than the two remaining bidders', the winner's resale mechanism is robust to the existence of further resale stages.

[39] He can obtain this optimal payoff by immediately dropping out.

(iii) H/F is nondecreasing;
(iv) $\frac{d}{dv}\frac{H}{F}(d) > 0$.

(iii) is the assumption of reverse-hazard-rate stochastic dominance of H –the distribution of bidder n's value–over F–the common distribution of bidders 1 to $n-1$'s values (from (i)): bidders 1 to $n-1$ are homogeneous "weak" bidders, while bidder n is "strong." From (ii), the reverse hazard rate of H is nondecreasing, that is, H is logconvex. (iv) rules out the simple symmetric case where $F = H$ (and where the PBE is equivalent to the equilibrium with no resale allowed).

Here, a PBE under the FD regime is obtained by applying the randomization procedure to the PBE under a "partial disclosure," or PD, regime, where only the bids from bidders 1 to $n-1$ are publicly disclosed. We look for a pure PBE where bidders 1 to $n-1$ follow the same bidding function β and bidder $n+1$ follows a bidding function δ such that $\beta \geq \delta$; $\beta(c) = \delta(c) = c$; $\beta(d) = \delta(d)$; and the derivatives β' and δ' exist and are strictly positive over $(c, d]$. Let α and γ be the inverses of β and δ. We denote $\rho(v, w)$, with $v \leq w$, the optimal resale price $\rho^s(v, w)$ defined as in (1) with $F_1 = F$ and $F_2 = H$ and we denote r the function $\rho(\alpha, \gamma)$.

Assume bidder 1, for example, with value v_1 wins with the bid b_1 such that $\gamma(b_1) > v_1$ and observes that bidder j with bid b_j is the highest bidder among bidders $2, ..., n-1$. If $\alpha(b_j) \leq v_1$, he offers to resell to bidder n at the price $\rho(v_1, \gamma(b_1))$[40]. Otherwise, he implements his Myerson mechanism by making sequential make-it-or-leave-it offers: first to bidder n at the price $\rho(\alpha(b_j), \gamma(b_1))$, and, if bidder n refuses, next to bidder j at the price $\alpha(b_j)$[41]. If profitable resale is possible, bidder n upon winning resells to the bidder who has submitted the second highest bid b at the price $\alpha(b)$.

As we show in Appendix 8, the FOC's can be written as the following

[40] This mechnaism is robust to further resale.

[41] If further resale is allowed, bidder 1 resells at the price $\alpha(b_j) H(\rho(\alpha(b_j), \gamma(b_1))) + \rho(\alpha(b_j), \gamma(b_1))(H(\gamma(b_1)) - H(\rho(\alpha(b_j), \gamma(b_1))))$ to bidder j, who will then offer to resell to bidder n at the price $\rho(\alpha(b_j), \gamma(b_1))$.

system of differential equations:

$$\begin{aligned}
& \frac{d}{db} \ln H(\gamma(b)) \\
= & \frac{1}{r(b)-b} \\
& \left\{1-\frac{n-2}{n-1} \frac{(\alpha(b)-b) H(r(b))+(r(b)-b)(H(\gamma(b))-H(r(b)))}{(r(b)-b) H(\gamma(b))}\right\} ; (19)
\end{aligned}$$

$$\frac{d}{db} \ln F(\alpha(b))=\frac{1}{(n-1)(r(b)-b)} . (20)$$

Moreover, these FOC's, along with the boundary conditions $\beta(c)=\delta(c)=c, \beta(d)=\delta(d)$, the strict monotonicity of β, δ, and the inequality $\beta \geq \delta$ are sufficient for an equilibrium. We also show in Appendix 8 that bidder n with value v is indifferent among all bids in $[c, r^{-1}(v)]$.

Although the FOC's are the same as in a common-value model[42], the literature on this model is less useful here, since it does not provide explicit expressions for the equilibrium strategies[43]. It is rather the methods developed for the FPA without resale and with heterogeneous bidders that allow us to prove Theorem 4 (i,ii) below. The proof in Appendix 9 of the existence in (i) proceeds by studying the solution to the system (19,20) with initial condition $\alpha(d)=\gamma(d)=\eta$, where η is a parameter such that $\eta<d$ and which, when the inverses β, δ form a PBE, is the bidders' common bid at d . Because the system (properly rewritten) is locally Lipchitz at such an initial condition, the standard theory of ordinary differential equations applies and implies that any solution, where defined, is strictly monotonic, and such that $\frac{H(\gamma(b))}{F(\alpha(b))}$ is nondecreasing and $\alpha \leq \gamma$. Moreover, the functions α, γ and the lower extremity $\underline{b}(\eta)$ of their largest definition interval are monotonic with respect to η. We then prove that there exist some values η of the parameter

[42] Where the common value as a function of the signals is $\rho\left(\max_{1 \leq i \leq n-1} v_i, \max_{1 \leq i \leq n} v_i\right)$.

[43] From (19,20), if the bid distributions were identical, F and H would be equal, which is impossible by assumption (iv).

such that $\underline{b}(\eta) > c$ and others such that $\underline{b}(\eta) < c$. Finally, we show that there exists an intermediate value of the parameter such that $\underline{b}(\eta) = c$ and, the remaining sufficient condition for a PBE, $\alpha(c) = \gamma(c) = c$ is satisfied. To this end, we rule out jumps, due to small decreases of η, of the graphs of the functions α, γ from common points on the 45-degree line, where they end up when $\underline{b}(\eta) > c$, to points to the vertical of and away from (c, c).

To prove the uniqueness in (i), we transform in Appendix 9 the system (19,20) into a differential system in $\varphi = \gamma\beta$ and β. We then show that, if there existed two PBE's, the function φ that would correspond to the higher value of the parameter η would be smaller. Then, the value $\beta(d)$, through its positive relation with φ (obtained by integrating (20)) would also be smaller, which would contradict the initial condition $\beta(d) = \eta$.

In Appendix 8, we also prove Theorem 4 (iii). The randomization procedure constructs an "equivalent" behavioral bidding strategy for bidder n and keeps the other bidders' bidding strategies unchanged. Under FD, after bidder 1, for example, with value v_1 wins and observes that bidder n's b_n is the second highest bid, he offers to resell to bidder n at the price $\max(r(b_n), r(\beta(v_1)))$[44]. If bidder j, $2 \leq j \leq n-1$, has submitted the second highest bid b_j with $\alpha(b_j) > v_1$ bidder 1 (who must have deviated from his biding strategy) offers to resell first to bidder n at the price $r(b_j)$, and, if the offer is rejected, to bidder j at the price $\alpha(b_j)$[45]. If $\alpha(b_j) < v_1$, bidder 1 offers to resell to bidder n at the price $r(\beta(v_1))$.

Theorem 4:

(i) (existence and uniqueness under PD): Under PD, there exists one and only one PBE where the bidding functions β, δ are strictly increasing,

[44] Bidder 1 does not include the other bidders in his optimal mechanism because, according to the randomization procedure, bidder n's virtual value conditional on b_n is at least equal to $\alpha(b_n)$ and hence to the other bidders' values. This mechanism is robust to further resale.

[45] When further resale is allowed, the implementation of the optimal mechanism is similar to the one in Footnote 41.

such that $\beta(c) = \delta(c) = c, \beta(d) = \delta(d)$, $\beta \geq \delta$, and their inverses α, γ satisfy (19,20).

(ii) (stochastically larger bid from the strong bidder): At the PBE in (i), $\frac{H(\gamma(b))}{F(\alpha(b))}$ is nondecreasing over $(c, d]$.

(iii) (PBE under FD) The randomization procedure applied to the PBE in (i) produces an equivalent PBE under FD.

Proof: See Appendices 8 and 9.

The revenue comparisons in Corollary 2 also extend[46]. Let the function ψ be defined as in (15) with F and H instead of F_1 and F_2 and let it be differentiable at d. At a PBE with bidding functions β, δ as in Theorem 4, the eventual owner of the item is either the bidder with the highest value among bidders 1 to $n-1$ or bidder n, depending on whether v_n is smaller or larger than $\lambda(\max_{1i \leq n-1} v_i) = \rho(\max_{1i \leq n-1} v_i, \varphi(\max_{1i \leq n-1} v_i))$, where $\varphi = \gamma\beta$. Since $\lambda(\max_{1i \leq n-1} v_i) \geq \max_{1i \leq n-1} v_i$, the eventual owner is always among the two bidders with the highest values and is hence present at the last stage of the EA in all the PBE's from the previous subsection. In particular, the PBE where the function φ_w that defines the strategies at the last stage approximates ψ over $[w, d]$ if bidder n is one of the two remaining bidders and is the identity function otherwise brings more revenues[47]. On the other hand, a PBE constructed from a function φ_w that approximates a function further away from ψ than φ is brings less revenues[48].

8. Relations with the Literature

[46]To prove this extension, contrary to Theorem 4, we use our assumption of strict monotonicity of $v - \frac{1-F(v)}{f(v)}$.

[47]In fact, λ is different from ψ in every neighborhood of d since, from (19) and (20), $\varphi'(d) = \frac{f(d)}{h(d)}$, which is strictly smaller than 1 by our assumption (iv), and hence, from the definitions of ψ and λ, $\lambda'(d) = \frac{1+\varphi'(d)}{2} < 1 = \psi'(d)$.

[48]So also does the "extreme" PBE, where the two last remaining bidders follow an extreme PBE as in Subsection 3.4, and where the auction price is then the third highest value.

The common-value methods for the SPA were introduced and used by Milgrom (1981), Bikhchandani (1988), Bikhchandani and Riley (1991), Klemperer (1998), and Bulow *et al* (1999). In a common-value model, Mares (2005) obtains the optimal mechanism as an equilibrium of the SPA, which he selects among the multiplicity of equilibria of the SPA[49]. In a common-value model with two bidders and affiliated signals, Parreiras (2006) selects, among the infinity of equilibria of the SPA, the equilibrium that is the limit of equilibria of k-price auctions, as k tends towards zero[50]. The FPA with common value is studied in Wilson (1969, 1977), Ortega-Reichert (1968), Rothkopf (1969), Reece (1978), Milgrom (1979 a, b), Milgrom and Weber (1982 a); and symmetry (of at least some degree) at the equilibrium of asymmetric common-value FPA's was encountered in Wilson (1967), Engelbrecht-Wiggans *et al* (1983), Hendricks and Porter (1988), Hendricks *et al* (1994), Campo *et al* (2003), Parreiras (2006), and Cheng and Tan (2008). The model of common value is a special case of the model of affiliation Milgrom and Weber (1982 b) introduced.

While resale brings an endogeneous positive externality, auctions with exogenous externalities are investigated in Jehiel *et al* (1996, 1999), Jehiel and Moldovanu (2000), and Das Varma (2002). Kamien *et al* (1989) consider, in a complete information model, Bertrand duopolists who compete for a contract and bargain, under the same procedures as in the present paper, for the terms of the subcontract.

Bikhchandani and Huang (1989) consider a common-value model with affiliated signals where the bidders are speculators who resell the item to final consumers with no private information. They and, in a more general model, Haile (1999) show that signalling during the SPA may prevent the

[49] If we applied Mares (2005)'s selection criterion to our model, we would obtain the revenue-superiority of the SPA over the FPA when the bidders are heterogeneous.

[50] If we applied this selection criterion to our model, we would obtain revenue-equivalence between the SPA and the FPA. Parreiras (2006) obtains the revenue-superiority of the SPA because the signals in his model are strictly affiliated, while here the values are independently distributed.

existence of a separating equilibrium under FD. Signalling at auctions also appeared in Ortega-Reichert (1968), Hausch (1986), Whaerer (1999), Goeree (2002), and Katzman and Rhodes-Kropf (2002). The ratchet effect is the cause of the nonexistence of a separating equilibrium of the FPA in Whaerer (1999), where the auctioneer and the auction winner engage in bargaining.

Under complete information throughout, Milgrom (1987), Campos e Cunha and Santos (1995), and Gale *et al* (2000) study resale in auctions. Ausubel and Cramton (1999) assume the post-auction resale to be efficient. In Haile (2000, 2001, 2003), bidders are uncertain about their own values. Haile (2000, 2001) assumes the values become public knowledge before resale. Haile (2003) considers the case where only the private uncertainty is lifted before resale and obtains expressions for the equilibria, conditional on their existence. Haile (2003) goes on to compare different auction formats and bargaining procedures at resale.

Gupta and Lebrun (1999) also assume that information becomes complete before resale and consider arbitrary exogeneous resale-price functions. Through the net values for winning, their model reduces to a common-value model. Due to the incomplete information at resale, the model of the present paper is not equivalent to such a model.

Cheng and Tan (2008) appeal to Hafalir and Krishna (2008)'s characterization of the equilibrium of the FPA with resale under ND and note that its bidding strategies form an equilibrium of a common-value model. In the present paper, we isolate (in Lemmas 1 and 3) the properties that our model shares with the common-value model and use them to prove our results, including explicit characterizations of PBE's under both disclosure regimes.

Garratt and Tröger (2006a) add to the bidders of a symmetric model one "speculator," whose has no consumption value for the item, and allow for discounting. They describe an infinite class of PBE's of the SPA and one PBE of the FPA. The results for the SPA are extended in Garratt *et al* (2006b) to the two-bidder case where all biders have strictly positive values (they also describe n "extreme" PBE's in the n-bidder case). Garratt *et al* (2008) construct correlated equilibria of the EA with n bidders that dominate, from the bidders' point of view, the truth-bidding equilibrium.

Bose and Deltas (2002, 2004) show the presence of a winner's curse in auctions between one final consumer and several speculators who are not allowed to use the information from the auction. Pagnozzi (2007 a) shows that resale may occur at the equilibrium of a SPA that awards a project with random cost to one of two heterogenous bidders, one with limited liability and none with private information. Pagnozzi (2007 b) studies the effects of resale in a multi-unit auction when bidders have flat demands and information is complete.

Our model is closest to the model of Krishna (Chapter 4, 2005) and Hafalir and Krishna (2007, 2008), where resale follows an auction between two bidders with independent values that remain private. Krishna (Chapter 4, 2005) shows that, if the average values differ across bidders, there exists no pure PBE of the FPA under FD that results in an efficient final allocation. Hafalir and Krishna (2007, 2008) focus on the ND regime. Hafalir and Krishna (2008) obtain the same unique pure PBE's of the FPA we obtain for the two bargaining procedures at resale under this regime[51]. They show that, if the bidders are heterogeneous, the equilibrium of the FPA gives higher expected revenues than the truth-bidding equilibrium of the SPA. For classes of value distributions for which the equilibria of the FPA without resale can be characterized explicitly, Hafalir and Krishna (2007) compares the FPA's with and without resale.

Zheng (2002), in the standard independent-private-value model with n bidders, and Calzolari and Pavan (2006), in a discrete two-bidder model,

[51] In addition to the two ultimatum procedures Hafalir and Krishna (2008) consider "probabilistic" procedures, where the resale-price maker is chosen at random according to exogeneous probabilities. From our Corollary 3, the auctioneer's revenues are increasing in the probability with which the auction winner is the resale-price maker.

design optimal mechanisms when resale is possible[52].

9. Conclusion

In the independent private value with 2 bidders, we gave explicit formulas for PBE's of the FPA, SPA, and EA with resale under the ND and FD regimes. We proved our results by using three key properties our model shares with the common-value model. We designed the randomization procedure, which allows to circumvents the ratchet effect by constructing, for each pure PBE under ND, an equivalent behavioral PBE under FD. We compared the auctioneer's revenues across auctions and bargaining procedures at resale. We finally extended some of our results to n-bidder models.

Appendix 1

Lemma A1: *For all $b_i \geq c$ and v_i in $[c,d]$, the function $\widetilde{u}_i^s(v_j) = u_i^s\left(v_i, v_j; b_i, \beta_j(v_j)\right)$ of v_j is continuous at $v_j = \alpha_j(b_i)$ and almost all other v_j in $[c,d]$.*

Proof: From the definition of u_i^s and the continuity of ρ^s and α_i, $\widetilde{u}_i^s$ is continuous at v_j if $v_j \neq \rho_i^s(v_i, \max(v_i, \alpha_j(b_i)))$. Assume v_j is such that $v_j = \alpha_j(b_i)$ and $v_j = \rho_i^s(v_i, \max(v_i, \alpha_j(b_i)))$. Then, $v_j = v_i$. Since the function $\widetilde{u}_i^s$ always lies between v_i and v_j, it is continuous if $v_j = v_i$ and Lemma A1 follows. $||$

Proof of Lemma 2: **(i)**: Through the change of variables $w_j = \alpha_j(b_i)$, (8) implies

$$w_j \in \arg\max_{w_j' \in [c,d]} \int_c^{w_j} u_i^s\left(v_i, v_j; \beta_j\left(w_j'\right), \beta_j(v_j)\right) dF_j(v_j), \text{ (A1.1)}$$

for all w_j in $[c,d]$. For all w_j' in $[c,d]$, the objective function in (A1.1), as an integral, is absolutely continuous with respect to w_j. From Lemma A1 and

[52] Here, we focus on how resale changes particular auction procedures.

the continuity of f_j, the integrand is continuous with respect to v_j almost everywhere in $[c, d]$. Consequently, the derivative of the objective function at w_j exists and is equal to $u_i^s\left(v_i, w_j; \beta_j\left(w_j'\right), \beta_j\left(w_j\right)\right) f_j\left(w_j\right)$, for almost all w_j in $[c, d]$. Since u_i^s and f_j are bounded, the assumptions of a variant[53] of Theorem 2 in Milgrom and Segal (2002) are satisfied. From this variant and the change of variables $w_j = \alpha_j\left(b_i\right)$, (i) follows for all b_i in $\left[c, \beta_j\left(d\right)\right]$.

(8) implies that the objective function is constant with respect to b_i' and b_i in $\left[\beta_j\left(d\right), +\infty\right)$. (i) then follows.

(ii): (ii) follows from (5, 6) and $\rho^s\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right) \leq \max\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right)$.

(iii): $u_i^s\left(v_i, \alpha_j\left(b\right); b, b\right)$ is, from (5-7), equal to $\rho_i^s\left(v_i, \alpha_j\left(b\right)\right)$ if $v_i < \alpha_j\left(b\right)$, to $\alpha_j\left(b\right)$ if $v_i = \alpha_j\left(b\right)$, and to $\min\left(v_i, \rho_j^s\left(\alpha_j\left(b\right), \max\left(\alpha_j\left(b\right), \alpha_i\left(b\right)\right)\right)\right)$ if $v_i > \alpha_j\left(b\right)$. (iii) follows. ||

Proof of Corollary 1: (i) Assume, for example, that bidder 2 is less aggressive at v, that is, $\beta_1\left(v\right) > \beta_2\left(v\right)$, or, equivalently, $\varphi\left(v\right) > v$ (the proof is similar in the other case). Since bidder 2 with value v wins the auction only if bidder 1's value is smaller than v, no trade occurs after bidder 2 wins. Let b be a bid in $(\beta_1(\varphi^-\left(v\right)), r^{-1}\left(v\right))$. Then, $\alpha_1\left(b\right) < \alpha_2\left(b\right)$ and $r\left(b\right) < \min\left(v, \alpha_2\left(b\right)\right)$.

By continuity, for all v_1 in a neighborhood of $\alpha_1\left(b\right)$, $\rho^s\left(v_1, \alpha_2\left(\beta_1\left(v_1\right)\right)\right)$ is smaller than v and $\alpha_2\left(b\right)$ and hence is equal to both net values $u_2^s\left(v, v_1; \beta_1\left(v_1\right), \beta_1\left(v_1\right)\right)$ and $u_2^s\left(\alpha_2\left(b\right), v_1; \beta_1\left(v_1\right), \beta_1\left(v_1\right)\right)$. Consequently, from (9) and (10), the first-order effect[54] of a bid change from b on bidder 2's expected payoff when his value is v is the same as when his value is $\alpha_2\left(b\right)$. From the optimality of b for this latter value, this first-order effect vanishes at b. Since this is

[53] This is the variant (which can be proved as Theorem 2 in Milgrom and Segal 2002 from their Theorem 1) where the requirement that $f\left(x, .\right)$ be differentiable for all $x \in X$ is replaced by the requirement that $f\left(x^*\left(t\right), .\right)$ be differentiable, for any selection $x^*\left(.\right) \in X^*\left(.\right)$ and almost all $t \in (0, 1)$.

[54] From (9) and Lemma 1 (iii), the derivative with respect to $\alpha_1\left(b\right)$ of bidder 2's expected payoff in the SPA is $\left(u_2^s\left(v, \alpha_1\left(b\right); b, b\right) - b\right) f_1\left(\alpha_1\left(b\right)\right)$, which is equal to $\left(r\left(b\right) - b\right) f_1\left(\alpha_1\left(b\right)\right)$ in a neighborhood of b. From (10) and Lemma 1 (iii), the derivative with respect to b of the expected payoff in the FPA is $\left(u_2^s\left(v, \alpha_1\left(b\right); b, b\right) - b\right) \frac{d}{db} F_1\left(\alpha_1\left(b\right)\right) - F_1\left(\alpha_1\left(b\right)\right)$, equal to $\left(r\left(b\right) - b\right) \frac{d}{db} F_1\left(\alpha_1\left(b\right)\right) - F_1\left(\alpha_1\left(b\right)\right)$ in a neighborhood of b.

true for all b in the interval $(\beta_1(\varphi^-(v)), r^{-1}(v))$, bidder 2's expected payoff must be constant over the closure of this interval. Because it contains the equilibrium bid $\beta_2(v)$, all its elements are optimal bids.

(ii) For example, assume $\varphi(v_1) \geq v_1$ (the proof is similar for the reversed inequality). From the definitions of φ and λ^s_φ, $\lambda^s_\varphi(v_1) \leq \varphi(v_1) = \alpha_2\beta_1(v_1)$. If $v_2 < \lambda^s_\varphi(v_1)$, bidder 2 loses the auction and refuses bidder 1's resale offer. If $\lambda^s_\varphi(v_1) < v_2 < \varphi(v_1)$, bidder 2 loses the auction and accepts bidder 1's resale offer. If $\varphi(v_1) < v_2$, bidder 2 wins the auction and no mutually advantageous resale is possible. ||

Appendix 2

Lemma A2: $\frac{\partial_l}{\partial v_i}\rho^s(v,v) = \frac{1}{2}$, *for all* v *in* $(c,d]$.

Proof: Let v_1, v_2 be such that $c < v_i < v_j \leq d$, with $i \neq j$. Subtracting the definition (1) of $\rho^s(v_1,v_2)$ from v_j and dividing by $v_j - v_i$, we find:

$$1 = \frac{v_j - \rho^s(v_1,v_2)}{v_j - v_i}\left(1 + \frac{1}{f_j(\rho^s(v_1,v_2))}\frac{F_j(v_j) - F_j(\rho^s(v_1,v_2))}{v_j - \rho^s(v_1,v_2)}\right).$$

From the continuity of f_j at v_j and the continuity of ρ^s, $f_j(\rho^s(v_1,v_2))$ tends towards $f_j(v_j)$, when v_i tends towards v_j from below. Since the derivative of F_j at v_j is equal to $f_j(v_j)$, the limit of the ratio $\frac{F_j(v_j)-F_j(\rho^s(v_1,v_2))}{v_j-\rho^s(v_1,v_2)}$ is equal to $f_j(v_j)$. Consequently, the factor between parentheses in the equation above tends towards 2 and the lemma follows. ||

Proof of Lemma 2: **(i)** Since the sets of b_j's for which the formulas in C1-C2 apply are Borel sets and since C1 obviously defines a distribution, we only have to prove that the formula in C2 defines a probability distribution, that is, that $\exp\int_{r(b_j)}^{v_j}\frac{dw}{\alpha_i r^{-1}(w)-w}$ is nonincreasing in v_j over $[r(b_j), \varphi^+(\alpha_j(b_j))]$ and is equal to zero at $v_j = \varphi^+(\alpha_j(b_j))$, for all b_j such that $\alpha_j(b_j) > \alpha_i(b_j)$. The first result follows from the strict negativity of $\alpha_i r^{-1}(w) - w$, which itself follows from the strict inequality $\alpha_i(w) < r(w)$ over $\left[\beta_j(\varphi^-(\alpha_j(b_j))), \beta_j(\varphi^+(\alpha_j(b_j)))\right]$. Only the second result, which is equivalent to $\int_{r(b_j)}^{\varphi^+(\alpha_j(b_j))}\frac{dw}{\alpha_i r^{-1}(w)-w} = -\infty$, remains to be proved.

Since, as it is well known, $\int_{r(b_j)}^{\varphi^+(\alpha_j(b_j))} \frac{dw}{\varphi^+(\alpha_j(b_j))-w} = +\infty$, it suffices to prove that $\frac{\varphi^+(\alpha_j(b_j))-w}{\varphi^+(\alpha_j(b_j))-\alpha_i r^{-1}(w)}$, which is equal to $1/\left(\frac{w-\alpha_i r^{-1}(w)}{\varphi^+(\alpha_j(b_j))-w}+1\right)$, is bounded away from zero. Through a change of variables, this is equivalent to proving that $\frac{\varphi^+(\alpha_j(b_j))-r\beta_i(w)}{\varphi^+(\alpha_j(b_j))-w}$ is bounded away from zero when w tends towards $\varphi^+(\alpha_j(b_j))$ from below.

This last ratio is equal to the sum of the nonnegative ratio $\frac{\rho_i^s\left(w,\varphi^+(\alpha_j(b_j))\right)-\rho_i^s(w,\alpha_j\beta_i(w))}{\varphi^+(\alpha_j(b_j))-w}$ and the ratio $\frac{\varphi^+(\alpha_j(b_j))-\rho_i^s\left(w,\varphi^+(\alpha_j(b_j))\right)}{\varphi^+(\alpha_j(b_j))-w}$. From Lemma A2 above, the latter ratio tends towards $\frac{1}{2}$ as w tends towards $\varphi^+(\alpha_j(b_j))$. Consequently, the sum is bounded away from zero and C2 defines a probability distribution[55].

From C2, the derivative with respect to v_j of the function $F_j(v_j|b_j)$ is equal to $\frac{1}{v_j-\alpha_i r^{-1}(v_j)} \exp \int_{r(b_j)}^{v_j} \frac{dw}{\alpha_i r^{-1}(w)-w}$, for all v_j in $[r(b_j), \varphi^+(\alpha_j(b_j)))$. It is bounded in any closed subinterval of this semiopen interval. Moreover, we have already proved that $F_j(v_j|b_j)$ is continuous at the upper extremity $\varphi^+(\alpha_j(b_j))$ (where it is equal to 1). Consequently, it is absolutely continuous.

(ii) Let F_j^* be the marginal distribution of values of the joint distribution of values and bids obtained from the marginal distribution $F_j\alpha_j$ of bids and the conditional distribution $F_j(.|.)$ of values on bids, defined, in C1-C2. We want to prove $F_j^* = F_j$.

If $\beta_j(v_j) \geq \beta_i(v_j)$, then $F_j(v_j|b_j) = 1$, for all $b_j \leq \beta_j(v_j)$, and $F_j(v_j|b_j) = 0$, for all $b_j > \beta_j(v_j)$. From C1, these equalities are immediate if $\alpha_j(b_j) \leq \alpha_i(b_j)$. Assume $b_j \leq \beta_j(v_j)$ or, equivalently, $\alpha_j(b_j) \leq v_j$, and $\alpha_j(b_j) > \alpha_i(b_j)$, which implies $\beta_i(\alpha_j(b_j)) > \beta_j(\alpha_j(b_j))$ (bidder i is more aggressive at $\alpha_j(b_j)$). Consequently, $\varphi^+(\alpha_j(b_j)) \leq v_j$. From C2, $F_j(v_j|b_j) = 1$. If $b_j > \beta_j(v_j)$, then $b_j > \beta_i(v_j)$ and $\alpha_i(b_j) > v_j$. If, moreover, $\alpha_j(b_j) > \alpha_i(b_j)$, we have $r(b_j) > \alpha_i(b_j) > v_j$. C2 again then implies $F_j(v_j|b_j) = 0$. Consequently, $F_j^*(v_j) = F_j\alpha_j\left(\beta_j(v_j)\right) = F_j(v_j)$.

Assume next $\beta_j(v_j) < \beta_i(v_j)$. From C2, if $v_j \leq r(b_j)$ and $\alpha_j(b_j) <$

[55] The proof for the other bargaining procedure (see Section 4) is similar and makes use of $\frac{\partial_r}{\partial v_j}\rho_j^b(\varphi^-(\alpha_i(b_i)), \varphi^-(\alpha_i(b_i))) = \frac{1}{2}$.

$\varphi^+(v_j)$, then $F_j(v_j|b_j) = 0$. The previous paragraph and $\beta_i(\varphi^+(v_j)) = \beta_j(\varphi^+(v_j))$ imply $F_j(v_j|b_j) = F_j(\varphi^+(v_j)|b_j) = 0$, for all $b_j > \beta_j(\varphi^+(v_j))$. Consequently,

$$F_j^*(v_j) = \int_{[c,r^{-1}(v_j)]} F_j(v_j|b_j)\, dF_j\alpha_j(b_j).$$

The previous paragraph and $\beta_i(\varphi^-(v_j)) = \beta_j(\varphi^-(v_j))$ imply $F_j(v_j|b_j) = 1$, for all $b \leq \varphi^-(v_j)$ and hence:

$$F_j^*(v_j) - F_j\left(\varphi^-(v_j)\right) = \int_{[\varphi^-(v_j),r^{-1}(v_j)]} F_j(v_j|b_j)\, dF_j\alpha_j(b_j). \text{(A2.1)}$$

From (i), the derivative with respect to v_j of $F_j(v_j|b_j)$ inside the integral in (A2.1) is not larger than $\frac{1}{v_j - \alpha_i r^{-1}(v_j)}$, which is bounded in a neighborhood of v_j (since $r\beta_j(v_j) > v_j$). Consequently, F_j^* is differentiable at v_j and the derivative can be taken under the integral sign, that is:

$$\frac{d}{dv_j}F_j^*(v_j) = \int_{[\varphi^-(v_j),r^{-1}(v_j)]} \frac{d}{dv_j}F_j(v_j|b_j)\, dF_j\alpha_j(b_j). \text{(A2.2)}$$

Again from (i), $\frac{d}{dv_j}F_j(v_j|b_j)$ in (A2.2) is equal to $\frac{1}{v_j-\alpha_i r^{-1}(v_j)} \exp \int_{r(b_j)}^{v_j} \frac{dw}{\alpha_i r^{-1}(w) - w}$, which is also equal to $\frac{1}{v_j-\alpha_i r^{-1}(v_j)}(1 - F_j(v_j|b_j))$. As a consequence,

$$F_j(v_j|b_j) + \left(v_j - \alpha_i r^{-1}(v_j)\right)\frac{d}{dv_j}F_j(v_j|b_j) = 1. \text{(A2.3)}$$

Integrating (A2.3) with respect to b_j according to $F_j\alpha_j$ over $[\varphi^-(v_j), r^{-1}(v_j)]$ and using (A2.1) and (A2.2), we find:

$$F_j^*(v_j) + \left(v_j - \alpha_i r^{-1}(v_j)\right)\frac{d}{dv_j}F_j^*(v_j|b_j) = F_j\alpha_j r^{-1}(v_j). \text{(A2.4)}$$

By definition of r, we have $v_j = \rho_i^s\left(\alpha_i r^{-1}(v_j), \alpha_j r^{-1}(v_j)\right)$ and hence:

$$F_j(v_j) + \left(v_j - \alpha_i r^{-1}(v_j)\right)\frac{d}{dv_j}F_j(v_j|b_j) = F_j\alpha_j r^{-1}(v_j)\,.\text{(A2.5)}$$

Subtracting (A2.5) from (A2.4), we find:

$$\frac{d}{dv_j}\left(F_j^*(w_j) - F_j(w_j)\right) = \frac{F_j(w_j) - F_j^*(w_j)}{w_j - \alpha_i r^{-1}(w_j)},\text{(A2.6)}$$

at $w_j = v_j$. Since $\beta_i(w_j) > \beta_j(w_j)$ and thus $r\beta_i(w_j) > w_j$, for all w_j in $\left(\varphi^-(v_j), \varphi^+(v_j)\right)$, the equality (A2.6) holds true everywhere in this interval. From (A2.1), we have:

$$F_j^*\left(\varphi^-(v_j)\right) = F_j\left(\varphi^-(v_j)\right).\ \text{(A2.7)}$$

Suppose $F_j^*(v_j) - F_j(v_j) > 0$. Then, from (A2.6), $\frac{d}{dv_j}\left(F_j^*(w_j) - F_j(w_j)\right)$ would be strictly negative everywhere in $\left(\varphi^-(v_j), v_j\right)$ and, since the cumulative functions are continuous from the right, $F_j^*\left(\varphi^-(v_j)\right) - F_j\left(\varphi^-(v_j)\right) > F_j^*(v_j) - F_j(v_j) > 0$, which contradicts (A2.7). The strict inequality $F_j^*(v_j) - F_j(v_j) > 0$ is similarly impossible. The equality $F_j^*(v_j) = F_j(v_j)$ follows.

(iii) Let $\widetilde{m}_j$ be the distributional strategy from (ii). Its support S is the closure of $\left\{(v_j, b_j) \in [c,d] \times \left[c, \beta_j(d)\right] | v_j \text{ belongs to the support of } F_j(.|b_j)\right\}$. From C1-C2, S is then as follows[56]:

$$S = \left\{\begin{array}{c}(v_j, b_j) \in [c,d] \times \left[c, \beta_j(d)\right] | \\ b_j = \beta_j(v_j) \text{ if } \beta_j(v_j) > \beta_i(v_j); \\ \text{and } b_j \in \left[\beta_j\left(\varphi^-\left(v_j^-\right)\right), r^{-1}(v_j)\right] \text{ if } \beta_j(v_j) \leq \beta_i(v_j)\end{array}\right\}\text{(A2.8)}$$

[56] S is also $\left\{\begin{array}{c}(v_j, b_j) \in [c,d] \times \left[c, \beta_j(d)\right] | \\ v_j = \alpha_j(b_j) \text{ if } \alpha_j(b_j) < \alpha_i(b_j); \\ \text{and } v_j \in \left[r(b_j), \varphi^+\left(\alpha_j(b_j)^+\right)\right] \text{ if } \alpha_j(b_j) \geq \alpha_i(b_j)\end{array}\right\}$, where $\varphi^+(v^+)$ is the right-hand limit $\lim_{w \to >v}\varphi^+(w)$ and is equal to $\inf\{w \in \{d\} \cup (v,d] \,|\varphi(w) = w\}$.

, where $\varphi^-(v^-)$ is the left-hand limit $\lim_{w \to_< v} \varphi^-(w)$ and is equal to $\sup \left\{ \begin{array}{c} w \in \{c\} \cup [c, v) \mid \\ \varphi(w) = w \end{array} \right\}$. Notice that, when $\beta_j(v_j) = \beta_i(v_j)$, $r^{-1}(v_j) = \beta_j(v_j)$.

Let $\widetilde{G}_j(.|.)$ be one of the regular conditional distributions such that, for all v_j, the support of $\widetilde{G}_j(.|v_j)$ is included in the section at v_j of the support S of $\widetilde{m}_j$ (such a conditional can be obtained by changing any conditional at all v_j in a measurable set of F_j-measure zero). We show that $\widetilde{G}_j(.|v_j) = G_j(.|v_j)$, for F_j-almost all v_j or, since F_j is absolutely continuous (with respect to the Lebesgue measure), for almost all v_j. Since φ^- is nondecreasing, and hence has at most a countable number of discontinuities, we may assume that φ^- is continuous at v_j, that is, in particular, $\varphi^-(v_j) = \varphi(v_j)$.

Assume v_j in (c, d) is such that $\beta_j(v_j) \geq \beta_i(v_j)$. From (A2.8), $\widetilde{G}_j(.|v_j)$ is concentrated at $\beta_j(v_j)$ if $\beta_j(v_j) > \beta_i(v_j)$. This is also the case if $\beta_j(v_j) = \beta_i(v_j)$, since then $r^{-1}(v_j) = \beta_j\left(\varphi^-\left(v_j^-\right)\right) = \beta_j(\varphi^-(v_j))$ (since v_j is a continuity point of φ^-) and $\beta_j(\varphi^-(v_j)) = \beta_j(v_j)$. From B1, $\widetilde{G}_j(.|v_j)$ is equal to $G_j(.|v_j)$.

Assume next v_j in (c, d) is such that $\beta_j(v_j) < \beta_i(v_j)$ and let w_j be in $(\varphi^-(v_j), \varphi^+(v_j))$. From $\beta_j(w_j) < \beta_i(w_j)$ and $\varphi^-(w_j) = \varphi^-(v_j)$, the section at w_j of S is $\left[\beta_j(\varphi^-(v_j)), r^{-1}(w_j)\right]$. From (i), $F_j(.|b_j)$ is absolutely continuous, for all b_j in $\left(\beta_j(\varphi^-(v_j)), r^{-1}(w_j)\right)$. Consequently, $1 - \widetilde{G}_j(b|w_j)$ is equal to $\int_{(b, r^{-1}(w_j))} f_j(w_j|b_j)\, dF_j\alpha_j(b_j) / f_j(w_j)$, for all b in $\left(\beta_j(\varphi^-(v_j)), r^{-1}(w_j)\right)$ and F_j-almost all w_j in $(\varphi^-(v_j), \varphi^+(v_j))$. Substituting its value from (i) to the density function $f_j(w_j|b_j)$, we find, from B2., $\widetilde{G}_j(b|w_j) = G_j(b|w_j)$. Since the expression (13) is obviously equal to 1 (and continuous from the left) at $b = r^{-1}(w_j)$ and is continuous from the right (the exponential is not larger than 1) at $b = \beta_j(\varphi^-(v_j))$, this equality holds true for all b in $\left[\beta_j(\varphi^-(v_j)), r^{-1}(v_j)\right]$.

There remains to prove that (13) defines a probability distribution over the specified support for all v_j that satisfies the condition of B2. Since the exponential is nonnegative, the expression (13) is obviously nondecreasing over $\left[\beta_j\left(\varphi^-\left(v_j\right)\right), r^{-1}\left(v_j\right)\right]$. We have already noticed that it is equal to 1 at the upper extremity of the specified support. Consequently, we only have to prove that it is nonnegative at $\beta_j\left(\varphi^-\left(v_j\right)\right)$. However, we have already proved that (13) defines a probability distribution (since it is equal to $\widetilde{G}_j\left(.|w_j\right)$) and hence that (13) is nonnegative at $\beta_j\left(\varphi^-\left(v_j\right)\right)$, for F_j-almost all w_j in $\left(\beta_j\left(\varphi^-\left(v_j\right)\right), r^{-1}\left(v_j\right)\right)$. Since (13) is continuous over this interval, its value at $\beta_j\left(\varphi^-\left(v_j\right)\right)$ must be nonnegative everywhere over it and, in particular, at v_j.

(iv) From C1, under the assumption of RS1 the distribution $F_j\left(.|b_j\right)$ is degenerate at $\alpha_j\left(b_j\right)$. If $v_i < \alpha_j\left(b_j\right)$, bidder i appropriates all the gains from trade by setting the resale price at $\alpha_j\left(b_j\right)$. If $w_j < v_i$, no mutually profitable resale exists and any price that results in no resale, such as v_i, is optimal. If $\beta_j\left(v_i\right) \leq \beta_i\left(v_i\right)$, we have $v_i \leq \alpha_j\beta_i\left(v_i\right)$ and hence $r\left(\beta_i\left(v_i\right)\right) = \rho_i^s\left(v_i, \alpha_j\beta_i\left(v_i\right)\right) \geq v_i$. The definition in RS1 then gives optimal resale prices when $\beta_j\left(v_i\right) > \beta_i\left(v_i\right)$ and when $\beta_j\left(v_i\right) \leq \beta_i\left(v_i\right)$ and $r\left(\beta_i\left(v_i\right)\right) \leq \alpha_j\left(b_j\right)$. If $\beta_j\left(v_j\right) = \beta_i\left(v_i\right)$, we have $r\left(\beta_i\left(v_i\right)\right) = v_i$ and RS1 defines optimal resale prices.

Assume $\beta_j\left(v_i\right) < \beta_i\left(v_i\right)$, that is, bidder i is more aggressive at v_i. The case that is left to examine in RS1 is $\alpha_j\left(b_j\right) < r\left(\beta_i\left(v_i\right)\right)$. In this case, $\alpha_j\left(b_j\right) < \alpha_j\beta_i\left(v_i\right) \leq \alpha_j\beta_i\left(\varphi^+\left(v_i\right)\right) = \varphi^+\left(v_i\right)$. Since $\alpha_j\left(b_j\right) \leq \alpha_i\left(b_j\right)$ and then $\beta_i\left(\alpha_j\left(b_j\right)\right) \leq b_j = \beta_j\left(\alpha_j\left(b_j\right)\right)$, we have $\alpha_j\left(b_j\right) \leq \varphi^-\left(v_i\right) \leq v_i$. No profitable resale is possible and $r\left(\beta_i\left(v_i\right)\right)$ is optimal.

From C2, under the assumption of RS2, $F_j\left(w_j|b_j\right)$ is equal to $1-\exp\int_{r(b_j)}^{v_j}\frac{dw}{\alpha_i r^{-1}(w)-w}$ over its support $\left[r\left(b_j\right), \varphi^+\left(\alpha_j\left(b_j\right)\right)\right]$. Computing the "conditional virtual value," the virtual value for this conditional distribution, we find, using (i):

$$\begin{aligned} & w_j - \frac{1-F_j\left(w_j|b_j\right)}{f_j\left(w_j|b_j\right)} \\ = \; & \alpha_i r^{-1}\left(w_j\right), \end{aligned}$$

for all w_j in $\left(r\left(b_j\right), \varphi^+\left(\alpha_j\left(b_j\right)\right)\right)$.

If $\beta_j(v_i) \leq \beta_i(v_i)$, $\delta_i(v_i; b_j) = \max(r(\beta_i(v_i)), r(b_j))$ is at least equal to v_i since $r(b_j) \geq v_i$. Consequently, it is optimal when there is no gain from trade. We may then assume that there exist positive gains from resale, that is, $v_i < \varphi^+(\alpha_j(b_j))$. The virtual conditional value at w_j is larger than v_i if and only if $w_j > \max(r(\beta_i(v_i)), r(b_j))$, and $\delta_i(v_i; b_j)$ is the unique optimal resale price.

Lastly, since $\max(v_i, r(b_j)) \geq v_i$, in order to prove the optimality of $\delta_i(v_i; b_j)$ in RS2 when $\beta_j(v_i) > \beta_i(v_i)$ (bidder j is more aggressive at v_i) we may also assume that gains from resale are possible, that is, that $v_i < \varphi^+(\alpha_j(b_j))$. In RS2, $\alpha_j(b_j) > \alpha_i(b_j)$ and hence $\beta_i(v_j) > b_j \geq \beta_j(v_j)$, for all v_j in $(\alpha_i(b_j), \alpha_j(b_j)]$, that is, bidder i is more aggressive over this interval. Consequently, $v_i < \varphi^-(\alpha_j(b_j)) \leq \alpha_i(b_j) \leq r(b_j)$ and, for all w_j in $(r(b_j), \varphi^+(\alpha_j(b_j)))$,

$$\begin{aligned} v_i &< \alpha_i(b_j) \\ &\leq \alpha_i r^{-1}(w_j). \end{aligned}$$

Consequently, $\max(v_i, r(b_j)) = r(b_j)$ is the unique optimal resale price.

(v) Assume first $\beta_j(v_i) \leq \beta_i(v_i)$. From B1, $G_i(.|v_i)$ is concentrated at $\beta_i(v_i)$ and hence $b_j \leq \beta_i(v_i)$. Then, $\delta_i(v_i; b_j) = r(\beta_i(v_i))$ is immediate in RS2. It is also immediate in RS1 when $\beta_j(v_i) = \beta_i(v_i)$, since then $b_j \leq \beta_i(v_i)$ implies $\alpha_j(b_j) \leq v_i = r(\beta_i(v_i))$. Under the assumption of RS1, assume $\beta_j(v_i) < \beta_i(v_i)$, that is, bidder i is more aggressive at v_i. From $b_j \leq \beta_i(v_i)$, we have $\alpha_i(b_j) \leq v_i$ and hence $\alpha_j(b_j) \leq v_i$. Moreover, $\alpha_j(b_j) \leq \alpha_i(b_j)$ implies $\beta_i(\alpha_j(b_j)) \leq b_j = \beta_j(\alpha_j(b_j))$. Consequently, $\alpha_j(b_j) \leq \varphi^-(\alpha_j(b_j)) = r(\beta_i(\varphi^-(\alpha_j(b_j)))) \leq r(\beta_i(v_i))$. The formula in RS1 then also gives $\delta_i(v_i; b_j) = r(\beta_i(v_i))$.

Assume next $\beta_j(v_i) > \beta_i(v_i)$, which implies $\alpha_i\beta_j(v_i) > v_i$ and $v_i < r\left(\beta_j(v_i)\right) = \rho_i^s\left(\alpha_i\beta_j(v_i), v_i\right)$, or, equivalently, $\alpha_j r^{-1}(v_i) < v_i$. From B2, the maximum of the support of $G_i(.|v_i)$ is $r^{-1}(v_i)$ and hence $b_j \leq r^{-1}(v_i)$.

Consequently, $\alpha_j(b_j) \leq \alpha_j r^{-1}(v_i) < v_i$ and the formula in RS1 implies $\delta_i(v_i; b_j) = v_i$. The inequality $b_j \leq r^{-1}(v_i)$ immediately implies that the formula in RS2 also gives $\delta_i(v_i; b_j) = v_i$. ||

Proof of Lemma 3: (i) From Lemma 2 (v), the resale price and hence the net value of the auction loser do not depend on his bid. (i.1) follows, which immediately implies (i.2).

(ii) If $\alpha_j(b) \geq \alpha_i(b)$, after bidder i with value $v_i = \alpha_i(b)$ wins a tie at b, he demands at resale $\delta_i(\alpha_i(b); b) = r(\beta_i(\alpha_i(b))) = r(b)$ (from Lemma 2 (ii) and $\beta_j(\alpha_i(b)) \leq b = \beta_i(\alpha_i(b))$). From C1-C2, $r(b)$ is the minimum of the support of the conditional distribution $F_j(v_j|b)$. Therefore, bidder j accepts this resale price with probability one and both bidders' net values are equal to $r(b)$.

(iii) From the definition of the net utility and, in RS1-RS2, of $\delta_i(v_i; b_j)$, $\int u_i^s(v_i, v_j; b, b)\, dF_j(v_j|b)$ is equal to:

(a) If $\alpha_j(b) > \alpha_i(b)$:

(a.1) If $v_i \leq \alpha_i(b)$: $r(b)$.

(a.2) If $\alpha_i(b) < v_i \leq r(b)$: $v_i F_i(r(\beta_i(v_i))|b) + r(\beta_i(v_i))(1 - F_i(r(\beta_i(v_i))|b))$.

(a.3) If $r(b) < v_i < \varphi^+(\alpha_j(b))$: $\int_{r(b)}^{v_i} v_j dF_j(v_j|b) + v_i(F_j(r(\beta_i(v_i))|b) - F_j(v_i|b)) + r(\beta_i(v_i))(1 - F_j(r(\beta_i(v_i))|b))$.

(a.4) If $\varphi^+(\alpha_j(b)) \leq v_i$: $\int v_j dF_j(v_j|b)$.

(b) If $\alpha_j(b) < \alpha_i(b)$:

(b.1) If $v_i \leq \alpha_j(b)$: $\alpha_j(b)$.

(b.2) If $\alpha_j(b) < v_i < r(b)$: v_i

(b.3) If $r(b) \leq v_i : r(b)$.

(c) If $\alpha_j(b) = \alpha_i(b)$: $\alpha_j(b)$.

Within the domains above, the expected net utility is a nondecreasing function of v_i. The expression in (a.3) is the value of the maximization problem below:

$$\max_p \int_{r(b)}^{v_i} v_j dF_j(v_j|b) + v_i(F_j(p|b) - F_j(v_i|b)) + p(1 - F_j(p|b)),$$

whose solution is $r(\beta_i(v_i)) \geq v_i$. It is then also equal to the value of the different maximization problem below:

$$\max_{p \geq v_i} \left\{ \int_{r(b)}^{p} \min(v_j, v_i)\, dF_j(v_j|b) + p(1 - F_j(p|b)) \right\}.$$

In fact, the two objectives functions coincide for $p \geq v_i$ and the latter objective function is not larger than the former for $p < v_i$. Since the latter objective function is nondecreasing in v_i, for all p, so is the value of the problem. The proof for (a.2) is similar (and simpler).

Moreover, the expressions above coincide at the boundaries of the different domains. The net expected utility is continuous in v_i and hence nondecreasing in v_i everywhere.

Reorganizing the different expressions above with respect to b, we find, when $\beta_j(v_i) \neq \beta_i(v_i)$, the values below for $\int u_i^s(v_i, v_j; b, b)\, dF_j(v_j|b)$:

(I) If $b \leq \beta_i(\varphi^-(v_i))$:

(I.1) If $\alpha_i(b) < \alpha_j(b)$: $\int v_j dF_j(v_j|b)$.

(I.2) If $\alpha_i(b) \geq \alpha_j(b)$: $r(b)$.

(II) When $\beta_j(v_i) < \beta_i(v_i)$:

(II.1) If $\beta_i(\varphi^-(v_i)) < b \leq r^{-1}(v_i)$: $\int_{r(b)}^{v_i} v_j dF_j(v_j|b) + v_i(F_j(r(\beta_i(v_i))|b) - F_j(v_i|b)) + r(\beta_i(v_i))(1 - F_j(r(\beta_i(v_i))|b))$.

(II.2) If $r^{-1}(v_i) < b < \beta_i(v_i)$: $v_i F_j(r(\beta_i(v_i))|b) + r(\beta_i(v_i))(1 - F_j(r(\beta_i(v_i))|b))$.

(II.3) If $\beta_i(v_i) \leq b \leq \beta_i(\varphi^+(v_i))$: $r(b)$.

(III) When $\beta_j(v_i) > \beta_i(v_i)$:

(III.1) If $\beta_i(\varphi^-(v_i)) < b \leq \beta_i(v_i)$: $\min(r(b), v_i)$.

(III.2) If $\beta_i(v_i) < b \leq \beta_i(\varphi^+(v_i))$: $\alpha_j(b)$.

(IV) If $\beta_i(\varphi^+(v_i)) < b$:

(IV.1) If $\alpha_i(b) < \alpha_j(b)$: $r(b)$.

(IV.2) If $\alpha_i(b) \geq \alpha_j(b)$: $\alpha_j(b)$.

When $\beta_j(v_i) = \beta_i(v_i)$, $\varphi^-(v_i) = \varphi^+(v_i) = v_i$ and only (I) and (IV) apply. The continuity with respect to b follows from the continuity within the domains above as well as the agreement among the definitions at the boundaries of their domains. ||

Proof of Theorem 2: Up to terms constant in b_i, bidder i's expected net payoffs can be written as follows:

$$\text{FPA: } \int_d^{b_i} \int u_i^s (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j)\, dF_j\alpha_j (b_j) - \int_c^{b_i} b_i dF_j\alpha_j (b_j) ;\text{(A2.9)}$$

$$\text{SPA: } \int_c^{b_i} \left(\int u_i^s (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j) - b_j \right) dF_j\alpha_j (b_j) .\text{(A2.10)}$$

From (12), in the FPA $F_j\alpha_j$ is continuously differentiable and hence, from Lemma 3 (iii), the derivative with respect to b of the first term in (A2.9) is the value at b of the integrand.

Proceeding as in the proof of Corollary 1, we obtain that the optimal bids in $\mathcal{E}$ of bidder i are also optimal under FD when bidder j follows his strategy in $\mathcal{E}'$. Since these bids form the supports of the bidding strategies $G_i (.|.)$, $i = 1, 2$, $\mathcal{E}'$ is a PBE under FD and Theorem 2 (i) is proved.

From (A2.10), Lemma 3 (i.2), (ii), and (iii), and the equality $r(b) = b$, any of a bidder's equilibrium bids in the SPA wins against bids that contribute nonnegatively to his net expected payoff and loses against those that would contribute nonpositively. Consequently, even if he was allowed to, a bidder would have no incentive to change his bid after learning his opponent's bid, and we have proved Theorem 2 (iv).

The final allocation in $\mathcal{E}'$ is the same as in $\mathcal{E}$. Assume, for example, that bidder 1's value v_1 is such that $\varphi(v_1) \geq v_1$. Then, bidder 1 bids $\beta_1(v_1)$ and $v_1 \leq \lambda_\varphi^s(v_1) = \rho^s(v_1, \varphi(v_1)) \leq \varphi(v_1)$. If $v_2 \leq \lambda_\varphi^s(v_1)$, B1-B2 imply that bidder 2 with value v_2 bids at most $\max(\beta_1(\varphi^-(v_1)), r^{-1}(v_2))$, which is not larger than $\beta_1(v_1)$. Consequently, neglecting ties, bidder 2 loses the auction and refuses bidder 1's offer. If $v_2 > \lambda_\varphi^s(v_1)$, bidder 2 accepts bidder 1's resale offer when bidder 1 wins and there is no profitable resale when bidder 2 wins.

Any bidder with the lowest possible value c obtains the same expected payoff–zero–in both PBE's. From Myerson (1981), the interim expected payoffs are then the same in $\mathcal{E}'$ as in $\mathcal{E}$. From the randomization procedure, the marginal bid distributions are the same. We have proved Theorem 2 (ii).

Since the strictly more aggressive, conditional on his value, bidder faces the same bid distribution in both PBE's and, from B1, submits the same bid, his expected payoff from the auction stage and his probability of winning are also the same. From Lemma 3 (v), he demands the same resale price, which is strictly larger than his value. In order to generate the same interim expected payoffs, the probability of resale must be the same in both PBE's and we have proved Theorem 2 (iii).

If the bids are not fully disclosed, a deviation from $\mathcal{E}'$ by a bidder after which he loses the auction has the same result as under FD, since, even under FD, the resale price the auction winner demands does not depend on the bid from the auction loser (from Lemma 3 (v)). A deviation after which he wins the auction is at most as profitable, since less information is available to make an optimal proposal at resale. Theorem 2 (v) follows. ||

Appendix 3

The PBE's of the SPA are somewhat independent of the value distribution of the equilibrium resale-price maker. When, for example, $\varphi(v) > v$ and the auction winner chooses the resale price, the bidding strategies remain part of a PBE if the probability distribution of bidder 1's value is changed in a neighborhood of v. In fact, in Theorem 1, the bids and the resale prices the bidders submit along the equilibrium path depend only on the values of the optimal resale price function ρ^s at (v_1, v_2) with $\varphi(v_1) = v_2$. However, $\rho^s(v_1, v_2)$, with $v_2 > v_1$, is bidder 1's optimal resale price and hence depends only on the distribution of bidder 2's value. The randomization procedure only uses the probability distribution of the equilibrium resale-price taker in order to transform his bidding function into a behavioral bidding strategy. Moreover, when $\varphi(v) = v$, whatever the value distributions are, both bidding strategies are pure and equal (to φ) at v. Corollary A3.1 below follows. In Corollary A3.1, we assume that the buyer(seller)-virtual-value functions are strictly increasing when the auction winner (loser) sets the resale price.

Corollary A3.1: *In the SPA where the auction winner (loser) sets the resale price, let $\mathcal{E}$ be a PBE as in Theorem 1 (modified as explained in Section 4). Then the bidding strategies in $\mathcal{E}$ remain part of a PBE when the bidders' values are distributed according to F_1' and F_2' if F_1' is equal to F_1 over $\{v_1 \in [c,d] \,|\varphi(v_1) < v_1\}$ $(\{v_1 \in [c,d] \,|\varphi(v_1) > v_1\})$ and F_2' is equal to F_2 over $\{v_1 \in [c,d] \,|\varphi(v_1) > v_1\}$ $(\{v_1 \in [c,d] \,|\varphi(v_1) < v_1\})$.*

As it can be easily checked, the results about the SPA go through when φ is only assumed to be nondecreasing, instead of strictly increasing and continuous. Corollary A3.2 below holds true when an inverse φ^{-1} of φ is a function such that v lies between the limits[57] of $\varphi(w)$ for w tending from below and from above to $\varphi^{-1}(v)$, for all v in $[c,d]$. Although φ may be constant or discontinuous in Corollary A3.2, β_1 and β_2 are strictly increasing and their inverses α_1 and α_2 are uniquely defined and continuous. Minor adjustments of some definitions in the previous proofs are necessary to carry over to such a more general φ. For example, Corollary 1 now applies to bidder 1 with value v such that[58] $\lim_{u \to_< v} \varphi(u) < v$, in which case the lower extremity $\varphi^-(v)$ of the set of optimal bids $[\varphi^-(v), v]$, over which bidder 1 randomizes under FD, is the largest fixed point w smaller than v of the correspondence $\Phi(w) = [\lim_{u \to_< w} \varphi(u), \lim_{u \to_> w} \varphi(u)]$.

[57] With the conventions $\lim_{w \underset{<}{\to} c} \varphi(w) = c$ and $\lim_{w \underset{>}{\to} d} \varphi(w) = d$.

[58] It applies to bidder 2 with value v such that $\lim_{u \to_< v} \varphi^{-1}(u) < v$.

Corollary A3.2[59]**:** *The results about the SPA hold true if the function* φ *from* $[c,d]$ *to* $[c,d]$ *is only required to be nondecreasing and if, in the formulas,* φ^{-1} *denotes an inverse of* φ.

We showed in Subsection 3.4 an application of Corollary A3.2 to the function φ such that $\varphi(v_1) = \theta^*$, for all v_1 in $[c, \theta^*]$, and $\varphi(v_1) = v_1$, for all v_1 in $[\theta^*, d]$. From Corollary A3.1, the bidding strategies of our behavioral PBE under FD for this function remain part of a PBE if F_1 is changed to any distribution F_1' (with increasing buyer virtual value)[60].

Appendix 4

We denote α_i^u the inverse of β_i^u. When $\beta_i^u = \beta_i^l$, we use the same notation β_i for both functions. Let (v', v'') be a maximum open interval where bidder i is more aggressive. Then, by continuity of β_i^u and β_j^u and the remark in the main text, we have $\beta_j^u(v') = \beta_i^u(v') = \beta_i^l(v') = \beta_j^l(v')$; $\beta_j^u(v'') = \beta_i^u(v'')$; and β_i^l and β_j^l are continuous at v'.

Lemma A4.1: For all v_i in (v', v''), $\delta_i(v_i; \beta_i(v_i)) = \alpha_j^u(\beta_i(v_i))$.

Proof: Suppose $\delta_i(v_i; \beta_i(v_i)) > \alpha_j^u(\beta_i(v_i))$. From Assumptions A6, A7.1, A7.2, and A7.3, there exists a neighborhood $(w', w'') \times (b', b'')$ of $(w, \beta_i(v_i))$ that is included in the interior of the support of bidder j's strategy and such that $v_j < \delta_i(v_i; b)$, for all (v_j, b) in this product (see Figure A1). From the uniqueness A2 of bidder i's optimal price, we find

$$v_j < \delta_i(v_i'; b),\text{(A4.1)}$$

[59] The supports of the bid distributions are not convex (intervals) when φ is constant or discontinuous. Under FD, a bidder's revised beliefs are not uniquely determined when he observes a bid in one of the gaps of his opponent's equilibrium bid distribution. Taking for these beliefs off the equilibrium path the particular distributions in our formulas, which depend on the (uniquely defined) inverses α_1 and α_2, ensures the equilibrium is perfect Bayesian.

[60] If F_1' is concentrated at c, they still remain part of a PBE and the alternative construction presented in Subsection 3.4 gives the equilibrium with no discounting in Garratt and Tröger (2006a).

for all (v_j, b) in $(w', w'') \times (b', b'')$ and all $v_i' \geq v_i$.

Consider the bids $\beta_i(v_i)$ and b in $(\beta_i(v_i), b'')$. From (A4.1), bidder j with value v_j in (w', w'') and any of these bids does not buy the item at resale. Since, being the less aggressive bider, he does not sell it neither, the probability that such a bidder j receives the item is the probability that he wins the auction. Obviously, this probability is higher with the bid b then with the bid $\beta_i(v_i)$. This contradicts probability invariance (the proposition in Subsection 3.5) and we have proved Lemma A4.1. ||

FIGURE A1

Lemma A4.2: *Let v_i be in (v', v'') and b' be in $(\beta_i^u(v'), \beta_i^u(v_i))$. If $(\delta_i(v_i; b'), b')$ belongs to the interior of the support of bidder j's distributional strategy, then $\frac{\partial \delta_i}{\partial b}(v_i; b') = 0$.*

Proof: Assume $(\delta_i(v_i; b'), b')$ belongs to the interior of the support of bidder j's distributional strategy. Then, $\alpha_j^u(b') < \delta_i(v_i; b')$ and, from Lemma A4.1, $\alpha_i(b') < v_i$.

Let $(w', w'') \times \left(b'', \widetilde{b}\right)$ be a neighborhood of $(\delta_i(v_i; b'), b')$ in the interior of bidder j's distributional strategy such that $\widetilde{b} < \beta_i^u(v_i)$ (which is possible;, since $b' < \beta_i^u(v_i)$). Suppose $\frac{\partial \delta_i}{\partial b}(v_i; b') = \sigma \neq 0$. If $\sigma > 0$, there exists $\underline{b}, \overline{b}$ such that $b'' < \underline{b} < b' < \overline{b} < \widetilde{b}$:

$$\begin{aligned}
& \delta_i\left(v_i; \overline{b}\right) \\
= \; & \delta_i(v_i; b') + \sigma\left(\overline{b} - b'\right) + o\left(\left|\overline{b} - b'\right|\right) \\
> \; & \delta_i(v_i; b') \\
> \; & \delta_i(v_i; b') + \sigma(\underline{b} - b') + o(|\underline{b} - b'|) \\
= \; & \delta_i(v_i; \underline{b}).
\end{aligned}$$

Consequently, there exist a subinterval $(\underline{w}, \overline{w})$ of (w', w'') such that (see Figure A2)

$$\delta_i\left(v_i; \overline{b}\right) > \overline{w} > \underline{w} > \delta_i(v_i; \underline{b}).$$

By continuity, there exists $\varepsilon' > 0$ such that $\underline{w} > \delta_i(w_i; \underline{b})$, for all $w_i < v_i + \varepsilon'$. If bidder j with value $w > \underline{w}$ bids $\underline{b}$, he receives the item when bidder i's value is smaller than $v_i + \varepsilon'$: when he does not win the auction, he accepts bidder i's resale offer. Consequently, $\Pr_j\left(w, \overline{b}\right) \geq F_i(v_i + \varepsilon')$. If he bids $\overline{b}$ instead, he loses the auction and refuses the resale price when bidder i's value is equal to or larger than v_i and hence $\Pr_j\left(w, \overline{b}\right) \leq F_i(v_i) < \Pr_j\left(w, \overline{b}\right)$, for all w in $(\underline{w}, \overline{w})$, which contradicts probability invariance. The proof for $\sigma < 0$ is similar. ||

FIGURE A2

Lemma A4.3: *For all v in (v', v''), there exists $\widetilde{b}$ in $[\beta_i(v'), \beta_i(v)]$, such that:*

(i)[61] *$\delta_i(v; b)$ is equal to $\alpha_j^u(b)$ for b larger than $\widetilde{b}$ and equal to $\alpha_j^u\left(\widetilde{b}\right)$ for b smaller than $\widetilde{b}$ and not smaller than $\beta_j^l\left(\alpha_j^u\left(\widetilde{b}\right)\right)$;*

(ii) For all $b < \beta_j^l\left(\alpha_j^u\left(\widetilde{b}\right)\right)$ such that profitable resale is possible, $\beta_j^l(\delta_i(v; b)) = b$.

(iii) $\delta_i(v; b)$ is nondecreasing in b over $(\beta_i(v'), \beta_i(v))$.

Proof: Lemmas A4.1 and A4.2 and continuity imply[62] that there exists $\widetilde{b}$ in $[\beta_i(v'), \beta_i(v)]$, such that $\delta_i(v; b)$ is equal to $\alpha_j^u(b)$ for b larger than $\widetilde{b}$ and equal to $\alpha_j^u\left(\widetilde{b}\right)$ for b in $\left(\beta_i(v'), \widetilde{b}\right]$ and strictly larger than $\beta_j^l\left(\alpha_j^u\left(\widetilde{b}\right)\right)$. By continuity, if $\beta_j^l\left(\alpha_j^u\left(\widetilde{b}\right)\right) > \beta_i(v')$, then $\delta_i\left(v, \beta_j^l\left(\alpha_j^u\left(\widetilde{b}\right)\right)\right) = \alpha_j^u\left(\widetilde{b}\right)$.

A simple application of the mean value theorem shows that, when decreasing b, once $(\delta_i(v; b), b)$ reaches the graph of β_j^l, then it does not go back in the interior of the support. (ii) follows. (iii) is a consequence of (i) and (ii). ||

[61] If $\widetilde{b}$ is one of the extremities of $[\beta_i(v'), \beta_i(v)]$, then only one of of the statements apply over the interval $(\beta_i(v'), \beta_i(v))$.

[62] It is simple to prove that, when decreasing b, once $(\delta_i(v; b), b)$ has left the graph of β_j^u it does not come back. It follows from the strict monotonicity of β_j^u and the independence of $\delta_i(v; b)$, when in the interior, with respect to b.

Lemma A4.4: *For all v_i in (v', v''):*

(i) there exists no $\varepsilon > 0$ such that $\delta_i(v_i; b)$ is equal to $\alpha_j^u(b)$, for all b in $(\beta_i(v_i) - \varepsilon, \beta_i(v_i))$;

(ii) $\delta_i(v_i; b) = \alpha_j^u \beta_i(v_i)$, for all b in $\left[\beta_j^l\left(\alpha_j^u \beta_i(v_i)\right), \beta_i(v_i)\right]$.

Proof: **(i)** Suppose there exists $\varepsilon > 0$ such that, for all b in $(\beta_i(v_i) - \varepsilon, \beta_i(v_i))$, $\delta_i(v_i; b) = \alpha_j^u(b)$. Let b be in $(\beta_i(v_i) - \varepsilon, \beta_i(v_i))$ (see Figure A3). If bidder j with value $v_j = \alpha_j^u(b)$ submits $b' > b$, he will not accept any resale offer, since, for all $w_i \geq \alpha_i(b')$, $\delta_i(w_i; b') \geq \delta_i(\alpha_i(b'); b') = \alpha_j^u(b')$ (from Lemma A4.1). Consequently, his expected payoff comes only from the auction stage. We obtain the FOC's:

$$\text{FPA} \quad : \quad \left[\frac{d}{db}(v_j - b) F_i(\alpha_i(b))\right]_{v_j = \alpha_j^u(b)} \leq 0. \text{ (A4.2)}$$

$$\text{SPA} \quad : \quad \left(\alpha_j^u(b) - b\right) \frac{d}{db} F_i(\alpha_i(b)) \leq 0 \text{ (A4.3)}.$$

Since, when profitable resale are possible $\delta_i(v, b')$ is nondecreasing in v, we have

$$\delta_i(v, b') = \alpha_j^u(b'),$$

for all b' in $(\beta_i(v_i) - \varepsilon, \beta_i(v_i))$ and all $v < v_i$ such that resale is possible. Let $w_j > \alpha_j^u(b)$ be such that the couple (w_j, b) belongs to the interior of the support of bidder j's strategy. Then, the expected payoff $P_j(w_j; b')$ of bidder j with value w_j if he submits b' in $(\beta_i(v_i) - \varepsilon, b]$ is equal to:

$$\begin{aligned}
& P_j(w_j; b') = \\
\text{FPA:} \quad & (w_j - b') F_i(\alpha_i(b')) \ \text{ SPA: } \int^{b'} (w_j - b_i)\, dG_i(b_i) \\
& + \left(w_j - \alpha_j^u(b')\right)\left(F_i\left(\alpha_i\left(\min\left(\beta_j^u(w_j), \beta_i(v_i)\right)\right)\right) - F_i(\alpha_i(b'))\right) \\
& + \int_{\min\left(\beta_j^u(w_j), \beta_i(v_i)\right)}^{\beta_i(w_j)} \max\left(w_j - \delta_i(\alpha_i(b''); b'), 0\right) dF_i \alpha_i(b''). \quad \text{(A4.4)}
\end{aligned}$$

In fact, no resale is possible after bidder i wins by bidding above $\beta_i(w_j)$.

Since, from Lemma A4.3, the last term in (A4.4) is nonincreasing in b', the derivatives of the first two terms give an upper bound on the rate of increase of $P_j(w_j; b')$. We find:

$$\begin{aligned}
& \limsup_{\varsigma \to >0} \frac{P_j(w_j; b) - P_j(w_j; b-\varsigma)}{\varsigma} \leq \\
\text{FPA} \quad : \quad & \left[\frac{d}{db}(v_j - b) F_i(\alpha_i(b))\right]_{v_j = \alpha_j^u(b)} \quad \text{SPA: } \left(\alpha_j^u(b) - b\right) \frac{d}{db} F_i(\alpha_i(b)) \\
& - \left(F_i\left(\alpha_i\left(\min\left(\beta_j^u(w_j), \beta_i(v_i)\right)\right)\right) - F_i(\alpha_i(b'))\right) \frac{d}{db}\alpha_j^u(b) \\
< \quad & 0,
\end{aligned}$$

from (A4.2, A4.3). Consequently, bidder j with value w_j is better off deviating from b, which cannot occur at an equilibrium.

(ii): (ii) follows immediately from (i) and Lemma A4.3. ||

FIGURE A3

Lemma A4.5: *For al v in (v', v''):*
(i) $\frac{d}{dv}\beta_j^l\left(\alpha_j^u\left(\beta_i^u(v_i)\right)\right) = 0$;
(ii) $\beta_j^l(v) = \beta_j^l(\underline{v})$;
(iii) $\delta_i(v; b) = \alpha_j^u(\beta_i(v))$, *for all* $b \leq \beta_i(v)$ *such that strictly (mutually) profitable resale is possible.*

Proof: **(i)** Assume $\frac{d}{dv}\beta_j^l\left(\alpha_j^u\left(\beta_i^u(v_i)\right)\right) > 0$ (see Figure A4). Then, there exists $v_j < \alpha_j^u\left(\beta_i^u(v_i)\right)$ such that $\beta_j^l(v_j) < \beta_j^l\left(\alpha_j^u\left(\beta_i^u(v_i)\right)\right)$. From Lemma 4.3 (ii), for all b in $\left(\beta_j^l(v_j), \beta_j^l\left(\alpha_j^u\left(\beta_i^u(v_i)\right)\right)\right)$, $(\delta_i(v_i; b), b)$ belongs to the graph of β_j^l. Then, β_j^l is strictly increasing over $\left(v_j, \alpha_j^u\left(\beta_i^u(v_i)\right)\right)$, otherwise $\delta_i(v_i; b)$ could not be continuous in b, contradicting Assumption A7.3; and $\delta_i(v_i; b)$ is strictly increasing in b, otherwise β_j^l could not be continuous, contradicting Assumption A7.2[63]. From the mean value theorem and Assumption A7.2, there exists w_j in $\left(v_j, \alpha_j^u\left(\beta_i^u\left(v_i\right)\right)\right)$ such that $\frac{d}{dv}\beta_j^l\left(w_j\right) > 0$.

From Lemma A4.4 (ii), the expected payoff $P_j\left(w_j; b'\right)$ of bidder j with value w_j if he submits b' in $\left(\beta_j^l\left(w_j\right), \beta_j^u\left(w_j\right)\right)$ is equal to:

$$
\begin{array}{ll}
 & P_j\left(w_j; b'\right) = \\
\text{FPA:} & \left(w_j - b'\right) F_i\left(\alpha_i\left(b'\right)\right) \ \text{SPA:} \displaystyle\int^{b'} \left(w_j - b_i\right) dG_i\left(b_i\right) \\
 & + \displaystyle\int_{b'}^{\beta_j^u\left(w_j\right)} \left(w_j - \alpha_j^u\left(b''\right)\right) dF_i \alpha_i\left(b''\right).
\end{array}
$$

In fact, from Lemma A4.4 (ii), for all b'' in $\left(\beta_i\left(v'\right), \beta_j^u\left(v''\right)\right)$, bidder i with value $\alpha_i\left(b''\right)$ demands $\alpha_j^u\left(b''\right)$ after observing b in $\left(\max\left(\beta_i\left(v'\right), \beta_j^l\left(\alpha_j^u\left(b''\right)\right)\right), b''\right)$. We obtain the FOC's at $b = \beta_j^l\left(w_j\right)$:

$$
\begin{aligned}
\text{FPA} &: \left(\alpha_j^u\left(b\right) - b\right) \frac{d}{db} F_i\left(\alpha_i\left(b\right)\right) - F_i\left(\alpha_i\left(b\right)\right) \leq 0. \ \text{(A4.5)} \\
\text{SPA} &: \left(\alpha_j^u\left(b\right) - b\right) \frac{d}{db} F_i\left(\alpha_i\left(b\right)\right) \leq 0 \ \text{(A4.6)}.
\end{aligned}
$$

There exists $\varepsilon, \varepsilon' > 0$ such that, for all b' in $\left(\beta_j^l\left(w_j\right) - \varepsilon', \beta_j^l\left(w_j\right)\right)$, the

[63] Since, according to $F_j\left(.|b\right)$, there is zero probability that bidder j's value is strictly larger than the value $\alpha_j^l\left(b\right)$ of the inverse of β_j^l at b, $\alpha_j^l\left(b\right)$ must be a mas point of $F_j\left(.|b\right)$ in order to be an optimal resale price for bidder i.

expected payoff $P_j(w_j; b')$ is:

$$
\begin{aligned}
&P_j(w_j; b') = \\
\text{FPA:}\quad &(w_j - b') F_i(\alpha_i(b')) \ \text{ SPA: } \int^{b'} (w_j - b_i)\, dG_i(b_i) \\
&+ \int_{b'}^{\beta_j^l(w_j)+\varepsilon} \left(w_j - \alpha_j^u(b'')\right)\, dF_i\alpha_i(b''), \\
&+ \int_{\beta_j^l(w_j)+\varepsilon}^{\beta_i(w_j)} \max\left(w_j - \delta_i(\alpha_i(b''); b'), 0\right) dF_i\alpha_i(b'') \\
&+ \int_{\beta_i(w_j)}^{\beta_i\left(\alpha_j^u(\beta_i^u(v_i))\right)} \left(w_j - \alpha_j^l(b')\right) dF_i\alpha_i(b'').
\end{aligned}
$$

as, from Lemma A4.4 (ii), $\delta_i(\alpha_i(b''), b') = \alpha_j^u(b'')$, for all b'' in a neighborhood of $\beta_j^l(w_j)$, and, from the first paragraph of the present proof, $\delta_i(v_i, b') = \alpha_j^l(b')$ and hence $\delta_i(v_i', b') = \alpha_j^l(b')$ for all $v_i' > v_i$, in a particular for v_i' in $\left(w_j, \alpha_j^u(\beta_i^u(v_i))\right)$.

Since, from Lemma A4.3, the third term above is nonincreasing in b', we find, at $b = \beta_j^l(w_j)$:

$$
\begin{aligned}
&\limsup_{\varsigma \to > 0} \frac{P_j(w_j; b) - P_j(w_j; b - \varsigma)}{\varsigma} \leq \\
\text{FPA} \quad : \quad &\left(\alpha_j^u(b) - b\right) \frac{d}{db} F_i(\alpha_i(b)) - F_i(\alpha_i(b)) \ \text{ SPA: } \left(\alpha_j^u(b) - b\right) \frac{d}{db} F_i(\alpha_i(b)) \\
&- \left(F_i\left(\alpha_j^u(\beta_i(v_i))\right) - F_i(w_j)\right) \frac{d}{db} \alpha_j^l(b) \\
< \quad &0,
\end{aligned}
$$

from (A4.5, A4.6). Consequently, bidder j with value w_j is better off deviating from $\beta_j^l(w_j)$ (to a lower bid), which cannot occur at an equilibrium.

(ii) From the remark in the main text, β_j^l is continuous at v'. For all v_j in (v', v''), $v_i = \alpha_i \beta_j^u(v_j)$ belongs to (v', v'') and, from (i), $\frac{d}{dv}\beta_j^l(v_j) = 0$. Consequently, β_j^l is constant and equal to $\beta_j^l(v')$ over $[v', v'')$.

(iii) The section at $b < \beta_j^l(v')$ of the support of bidder j's strategy is included in $[c, v']$. Consequently, there is no profitable resale after bidder i with value v_i in (v', v'') (for almost all such v_i) wins with a bid below $\beta_j^l(v')$. ||

FIGURE A4

Lemma A4.6: *For all j, define the function β_j^* as follows:*

$$\beta_j^*(w_j) = G_j^{-1} F_j(w_j),$$

where G_j is the cumulative function of the bidder j's marginal bid distribution. Then,

(i) When strictly profitable resale is possible for $b_j \leq \beta_i(v_i)$, $\delta_i(v_i; b_j)$ is equal to $\rho_i^s(v_i, \alpha_j^* \beta_i(v_i))$;

(ii) When bidder j is less aggressive at v_j, $\beta_j^*(v_j)$ belongs to the support of his bidding strategy conditional on v_j.

Proof: (i): Let v_i be such that bidder i is more aggressive at v_i. From Lemma A4.5 (iii), for all $b \leq \beta_i(v_i)$ such that profitable resale is possible, the resale price $\delta_i(v_i; b)$ bidder i demands after winning is equal to $\alpha_j^u(\beta_i(v_i))$. For all such b, we must then have:

$$\alpha_j^u(\beta_i(v_i)) \in \arg\max_p (p - v_i)(1 - F_j(p|b)), \text{(A4.7)}$$

. Since $\alpha_j^u(\beta_i(v_i)) \geq v_i$, (A4.7) then holds true for all $b \leq \beta_i(v_i)$. Consequently, we find:

$$\alpha_j^u(\beta_i(v_i)) \in \arg\max_p \int^{\beta_i(v_i)} (p - v_i)(1 - F_j(p|b))\, dG_j(b) . \text{(A4.8)}$$

Since α_j^u is strictly increasing in a neighborhood of $\beta_i(v_i)$ and the strategies are nondecreasing, we have $F_j(\alpha_j^u(\beta_i(v_i))|b) = 0$, for (almost-) all

$b > \beta_i(v_i)$. We find:

$$\begin{aligned} & \int^{\beta_i(v_i)} F_j(p|b)\, dG_j(b) \\ = & \int F_j(p|b)\, dG_j(b) \\ = & F_j(p). \end{aligned}$$

From (A4.8), we then have:

$$\alpha_j^u(\beta_i(v_i)) \in \arg\max_p (p - v_i)(G_j(\beta_i(v_i)) - F_j(p)).$$

From the definition of β_j^*, $G_j(\beta_i(v_i)) = F_j(\alpha_j^*(\beta_i(v_i)))$ and

$$\alpha_j^u(\beta_i(v_i)) \in \arg\max_p (p - v_i)\left(F_j(\alpha_j^*(\beta_i(v_i))) - F_j(p)\right).$$

Notice that[64] $G_j(\beta_i(v_i)) \geq F_j(\alpha_j^u(\beta_i(v_i)))$, because all $v_j < \alpha_j^u(\beta_i(v_i))$ bid less than $\beta_i(v_i)$. Consequently, $\delta_i(v_i; b) = \alpha_j^u(\beta_i(v_i)) = \rho_i^s(v_i, \alpha_j^* \beta_i(v_i))$ and (i) is proved.

(ii): Let $(\underline{v}, \overline{v})$ be the maximum open interval containing v_j where bidder i is more aggressive. From Lemma A4.5 (ii), β_j^l is constant over $[\underline{v}, \overline{v})$. Then, because bidder j bids higher than $\beta_j^l(\underline{v})$ for values in $(\underline{v}, d]$, we have $G_j(\beta_j^l(\underline{v})) \leq F_j(\underline{v})$. Thus, $G_j(\beta_j^*(v_j)) = F_j(v_j) > G_j(\beta_j^l(\underline{v}))$ and hence $\beta_j^*(v_j) > \beta_j^l(\underline{v})$.

Moreover, since bidder j does not bid higher than $\beta_j^u(v_j)$ for values smaller than v_j, we have $G_j(\beta_j^u(v_j)) \geq F_j(v_j)$. Consequently, $G_j(\beta_j^u(v_j)) \geq G_j(\beta_j^*(v_j))$ and hence $\beta_j^*(v_j) \leq \beta_j^u(v_j)$. ||

Lemma A4.7: (β_1^*, β_2^*) *can be completed into a PBE as in Theorem 1 of the FPA under ND that is equivalent to the original PBE under FD.*

Proof: If bidder i is more aggressive at v_i, then the derivative of bidder

[64] The inequality is actually strict.

i's expected payoff with respect to b at $b = \beta_i(v_i)$ is equal to:

$$
\begin{aligned}
&-G_j(\beta_i(v_i)) + \left(\alpha_j^u(\beta_i(v_i)) - \beta_i(v_i)\right)\frac{d}{db}G_j(\beta_i(v_i)) \text{ (FPA)},\\
&\left(\alpha_j^u(\beta_i(v_i)) - \beta_i(v_i)\right)\frac{d}{db}G_j(\beta_i(v_i)) \text{ (SPA)}
\end{aligned}
$$

since the resale price $\alpha_j^u(\beta_i(v_i))$ is accepted with probability one when bidder j has submitted $\beta_i(v_i)$. Since it must be equal to zero, we obtain:

$$
\begin{aligned}
\frac{d}{db}\ln G_j(\beta_i(v_i)) &= \frac{1}{\alpha_j^u(\beta_i(v_i)) - \beta_i(v_i)} \text{ (FPA)}\\
\alpha_j^u(\beta_i(v_i)) - \beta_i(v_i) &= 0 \text{ (SPA)}.
\end{aligned}
$$

From Lemma 4.6 (i), $\alpha_j^u(\beta_i(v_i)) = \rho_i^s\left(v_i, \alpha_j^*\beta_i(v_i)\right)$ and from the definition of β_j^*, $G_j = F_j\alpha_j^*$. Consequently:

$$
\begin{aligned}
\frac{d}{db}\ln F_j\left(\alpha_j^*(b)\right) &= \frac{1}{\rho_i^s\left(\alpha_i^*(b), \alpha_j^*(b)\right) - b} \text{ (FPA)},\\
\rho_i^s\left(\alpha_i^*(b), \alpha_j^*(b)\right) - b &= 0 \text{ (SPA)}
\end{aligned}
$$

at $b = \beta_i(v_i)$.

Similarly, by considering the derivative of bider j' payoff with respect to b at $(v_j, \beta_i(v_i))$ where v_j is in the support of $F_j(.|\beta_i(v_i))$, we find ($\beta_i(v_i) = \beta_i^*(v_i)$):

$$
\frac{d}{db}\ln F_i(\alpha_i^*(b)) = \frac{1}{\rho_i^s\left(\alpha_i^*(b), \alpha_j^*(b)\right) - b},
$$

at $b = \beta_i(v_i)$.

The same set of equations holds true when $\beta_1(v) = \beta_2(v) = b$, so that it holds over $\left(c, \overline{d}\right]$.

From Assumption A5, we have $\beta_1^*(c) = \beta_2^*(c) = c$ and $\beta_1^*(d) = \beta_2^*(d)$.

The hypotheses of Theorem 1 are satisfied and (β_1^*, β_2^*) can be completed into a PBE as in this theorem. This PBE is equivalent to the original PBE under FD. In fact, from the definition of β_1^*, β_2^*, the marginal bid distributions are the same; the more aggressive bidder follows the same bidding function; from Lemma A4.6 (i), the same resale prices are demanded when profitable resale is possible; and, from Lemma A4.6 (ii), the less aggressive bidder submits according to (β_1^*, β_2^*) a bid that belongs to the support of the original PBE. ||

Appendix 5

A Class of Examples Where the Equilibrium Bid Distributions Do not Increase When the Bargaining Power at Resale Goes From the Auction Loser to the Auction Winner:

Consider the example where there exists q in $(0,1)$ such that $F_1^{-1}(p) < F_2^{-1}(p)$, for all p in $(0,q]$, F_1 is strictly convex over $\left[0, F_1^{-1}(q)\right]$, and F_2 is strictly concave over $\left[0, F_2^{-1}(q)\right]$ (such an example obviously exists). Then, the definitions of the seller's and buyer's optimal resale price functions ρ^s and ρ^b readily imply $\rho^b\left(F_1^{-1}(p), F_2^{-1}(p)\right) > \frac{F_1^{-1}(p)+F_2^{-1}(p)}{2} > \rho^s\left(F_1^{-1}(p), F_2^{-1}(p)\right)$, for all p in $(0,q]$. If we denote by β_i^s and β_i^b the equilibrium bidding functions of the FPA under ND when the resale price is chosen by the auction winner and the auction loser, we find:

$$\begin{aligned}\beta_i^s(v) &= \frac{\int_0^{F_i(v)} \rho^s\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq}{F_i(v)} \\ &< \frac{\int_0^{F_i(v)} \rho^b\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq}{F_i(v)} = \beta_i^b(v),\end{aligned}$$

for all v_i in $\left(0, F_i^{-1}(q)\right)$. Consequently, for small bids, both bidders' bid distributions shift *downward* if the auction winner becomes the resale-price maker. The same conclusion applies to the PBE's under FD obtained through our randomization procedure.

Appendix 6

Lemma A6.1: *If ψ is differentiable over (c,d) and $F_1 \neq F_2$, then $\lambda^s_{F_2^{-1}F_1} \neq \psi$.*

Proof: The definitions of λ^s_φ and the equality $\varphi = F_2^{-1}F_1$ imply $\lambda^s_\varphi(v) = \rho^s\left(v, F_2^{-1}F_1(v)\right)$, or, equivalently:

$$\begin{aligned} & \lambda^s_\varphi(v) - \frac{1 - F_2\left(\lambda^s_\varphi(v)\right)}{f_2\left(\lambda^s_\varphi(v)\right)} \\ = \; & v - \frac{1 - F_1(v)}{f_1(v)} \\ & + (1 - F_1(v))\left(\frac{1}{f_1(v)} - \frac{1}{f_2\left(\lambda^s_\varphi(v)\right)}\right). \quad \text{(A6.1)} \end{aligned}$$

Assume that there exists v such that $\varphi(v) > v$. Suppose λ^s_φ and ψ are identical over $(\varphi^-(v), \varphi^+(v))$, where $\varphi^-(v)$ and $\varphi^+(v)$ are as in Subsection 2.3. From (15) and (A6.1), $f_1(w) = f_2(\psi(w))$ and $F_1(w) - F_2(\psi(w)) = f_1(w)(\psi(w) - w)$, for all w in $(\varphi^-(v), \varphi^+(v))$. Because its derivative then vanishes, $(F_1(w) - F_2(\psi(w)))(\psi(w) - w)$ is constant over this interval. However, this is impossible since it tends towards zero at the extremities, while being strictly positive in the interior (since $w < \lambda^s_\varphi(w) = \psi(w) < \varphi(w) = F_2^{-1}F_1(w)$). ||

Proof of Lemma 4: For simplicity, we drop the subscript $F_2^{-1}F_1$. If $F_2^{-1}F_1(v) = v$, then $\lambda^b(v) = \lambda^s(v) = v$.

Let v be such that $F_1(v) \neq F_2(v)$. Assume, for example, $F_2^{-1}F_1(v) > v$ (the proof when $F_2^{-1}F_1(v) < v$ is similar). From the definitions of λ^s, λ^b, we have $\lambda^s(v) = \rho^s\left(v, F_2^{-1}F_1(v)\right)$ and $v = \rho^b\left(F_1^{-1}F_2\left(\lambda^b(v)\right), \lambda^b(v)\right)$ and, consequently, $\lambda^s(v) < F_2^{-1}F_1(v)$ and:

$$F_1^{-1}F_2\left(\lambda^b(v)\right) < v; \text{(A6.2)}$$

$$v = \lambda^s(v) - \frac{F_1(v) - F_2(\lambda^s(v))}{f_2(\lambda^s(v))}; \quad (A6.3)$$

$$v = \lambda^b(v) + \frac{F_2\left(\lambda^b(v)\right) - F_1(v)}{f_1(v)}. \quad (A6.4)$$

(A6.3) is equivalent to (A6.1) in the previous proof. The equality (A6.5) below is the difference between (A6.4) and (A6.3).

$$\lambda^b(v) - \lambda^s(v) = \left(F_1(v) - F_2\left(\lambda^b(v)\right)\right)/f_1(v) - \left(F_1(v) - F_2(\lambda^s(v))\right)/f_2(\lambda^s(v)) . (A6.5)$$

If $\lambda^s(v) < \psi(v)$, then, from (A6.1) and (15), $f_1(v) > f_2(\lambda^s(v))$. Suppose the LHS of (A6.5) is nonnegative, that is, $\lambda^s(v) \leq \lambda^b(v)$. Then, from (A6.2), $F_1(v) - F_2(\lambda^s(v)) \geq F_1(v) - F_2\left(\lambda^b(v)\right) > 0$. Consequently, $\left(F_1(v) - F_2(\lambda^s(v))\right)/f_2(\lambda^s(v)) > \left(F_1(v) - F_2\left(\lambda^b(v)\right)\right)/f_1(v)$ and the RHS of (A6.5) is strictly negative, a contradiction. We have proved $\lambda^s(v) > \lambda^b(v)$ if $\lambda^s(v) < \psi(v)$. We can similarly prove $\lambda^s(v) < \lambda^b(v)$ if $\lambda^s(v) > \psi(v)$. Moreover, from (A6.5) and (A6.2): $\lambda^s(v) = \lambda^b(v)$ if and only if $f_1(v) = f_2(\lambda^s(v))$. From (A6.1), (15), and $v < d$ (since $F_1(v) \neq F_2(v)$)[65]. The lemma is proved. ||

Appendix 7

Corollary A7.1: *When ς in $(0,1)$ tends towards zero, the PBE of the $(\varsigma k_1, \varsigma k_2)$-PA tends towards the payoff-equivalent PBE, constructed from $\varphi = F_2^{-1}F_1^{k_1/k_2}$, of the SPA.*

Proof: In the $(\varsigma k_1, \varsigma k_2)$-PA, $\beta_1(v)$, for example, is, from (18), equal to:

$$\beta_1(v) = \frac{\int_c^v \rho^x\left(w, F_2^{-1}\left(F_1(w)^{k_1/k_2}\right)\right) dF_1(w)^{1/\varsigma k_2}}{F_1(v)^{1/\varsigma k_2}},$$

[65] In the case of different value upper extremities $d_1 < d_2$, we have $\lambda^s(d_1) = \psi(d_1) > \lambda^b(d_1)$ and Lemma 4 holds true for all v_1 in $[c, d_1)$.

for all v in $(c,d]$. Since the probability distribution $\left(\frac{F_1}{F_1(v)}\right)^{1/\varsigma k_2}$ over $[c,v]$ dominates $\left(\frac{F_1}{F_1(v)}\right)^{1/\varsigma' k_2}$, for all $0<\varsigma<\varsigma'$, $\beta_1(v)$ increases as ς decreases. When ς tends towards zero, the distribution $\left(\frac{F_1}{F_1(v)}\right)^{1/\varsigma k_2}$ converges weakly towards the degenerate distribution concentrated at v and, consequently, $\beta_1(v)$ tends towards $\rho^x(v,\varphi(v))$, where $\varphi = F_2^{-1}F_1^{k_1/k_2}$. Similarly, $\beta_2(v)$ increases and converges towards $\rho^x(\varphi^{-1}(v), v)$ when ς tends towards zero. ||

Appendix 8

The PD Regime

FOC for Bidders 1,...,n − 1

The derivative of the expected payoff of bidder 1 with value v in $(c,d]$ with respect to b in $(c,\beta(v))$ is:

$$
\begin{aligned}
&(\rho(v,\gamma(b))-b)\,F(\alpha(b))^{n-2}\frac{d}{db}H(\gamma(b))\\
&+\{vH(\rho(v,\gamma(b)))+\rho(v,\gamma(b))\,[H(\gamma(b))-H(\rho(v,\gamma(b)))]-b\}\frac{d}{db}F(\alpha(b))^{n-2}\\
&-H(\gamma(b))\,F(\alpha(b))^{n-2}. \text{ (A8.1)}
\end{aligned}
$$

Since $b<\beta(v)$, no profitable resale is possible with bidders 2 to $n-1$ and, as a consequence, bidder 1 offers the optimal resale price $\rho(v,\gamma(b))$ to bidder n. If bidder 1 loses, the winner of the auction will make no resale offer to him, since, if bidder 1 followed his strategy β, no profitable resale would be possible.

If bidder 1 increases his bid b by db, the first term above accounts for the event when the bids from bidders 2 to $n-1$ are smaller than b and the bid from bidder n is equal to b, that is, his value is $\gamma(b)$. Since $\gamma(b) \geq \rho(v, \gamma(b))$, bidder n accepts the resale offer, conditional on this event.

The second term accounts for the event where bidder n submits a bid smaller than b and the highest bid from bidders 2 to $n-1$ is b. Again no profitable resale is possible with bidders 2 to $n-1$, since their values is not larger than $\alpha(b) < v$. Bidder n accepts bidder 1's resale offer $\rho(v, \gamma(b))$ only if his value is at least as high, which occurs with conditional probability $H(\gamma(b)) - H(\rho(v, \gamma(b))) / H(\gamma(b))$.

From the envelope theorem, in the event that bidder 1 keeps winning after the bid raise, the first-order effect on his expected payoff due to the change of resale mechanism vanishes. The third term accounts for the increase in the expected payment at auction.

We then obtain the inequality below:

$$\begin{aligned}
&(r(b) - b) F(\alpha(b))^{n-2} \frac{d}{db} H(\gamma(b)) \\
&+ \left\{ \begin{array}{c} \alpha(b) H(r(b)) + \\ r(b)\left[H(\gamma(b)) - H(r(b))\right] - b \end{array} \right\} \frac{d}{db} F(\alpha(b))^{n-2} \\
&-H(\gamma(b)) F(\alpha(b))^{n-2} \geq 0. \quad \text{(A8.2)}
\end{aligned}$$

The derivative at $b > \beta(v)$ is:

$$\begin{aligned}
&\left\{ (\rho(v, \gamma(b)) - b) F(v)^{n-2} + \int_v^{\alpha(b)} (\rho(w, \gamma(b)) - b)\, dF(w)^{n-2} \right\} \frac{d}{db} H(\gamma(b)) \\
&+ \left\{ \begin{array}{c} (\alpha(b) - b) H(\rho(\alpha(b), \gamma(b))) + \\ (\rho(\alpha(b), \gamma(b)) - b)\left[H(\gamma(b)) - H(\rho(\alpha(b), \gamma(b)))\right] \end{array} \right\} \frac{d}{db} F(\alpha(b))^{n-2} \\
&-H(\gamma(b)) F(\alpha(b))^{n-2}. \quad \text{(A8.3)}
\end{aligned}$$

Let w be the highest value among bidders 2 to $n-1$'s, which bidder 1 deduces from their bids. His optimal resale auction consists in selling to bidder n if his value is larger than $\rho(\max(v, w), \gamma(b))$ and, otherwise, in selling to the bidder with the (highest) value w among bidders 2 to $n-1$ or in keeping the item, depending on whether w is larger than v or not[66].

The first term in (A8.3) above accounts for the event when bidders 2 to $n-1$ submit bids smaller than b and bidder n submits b. The second term for the event when the highest bid from bidders 2 to $n-1$ is b and bidder n has submitted a smaller bid[67]. Again, the first-order effect of a change of resale-mechanism in the event that bidder 1 keeps winning vanishes.

The expression (A8.3) where $v = \alpha(b)$ must then be nonpositive, that is, the reverse inequality in (A8.2) holds true. Consequently, the derivative of the expected payoff with respect to b at $v = \alpha(b)$ exists, is equal to the expression in (A8.2), and we obtain the FOC below:

$$\begin{aligned} & (r(b) - b)\frac{d}{db}\ln H(\gamma(b)) + \\ & (n-2)\left\{ \begin{array}{c} (\alpha(b) - b)\frac{H(r(b))}{H(\gamma(b))} + \\ (r(b) - b)\left[1 - \frac{H(r(b))}{H(\gamma(b))}\right] \end{array} \right\} \frac{d}{db}\ln F(\alpha(b)) \\ = \; & 1. \text{ (A8.4)} \end{aligned}$$

FOC for Bidder n

The derivative with respect to b of the expected payoff of bidder n with value v at $c < b < r^{-1}(v)$ is:

$$(\rho(\alpha(b), \gamma(b)) - b)\frac{d}{db}F(\alpha(b))^{n-1} - F(\alpha(b))^{n-1}.\text{(A8.5)}$$

[66] When $w > v$ and further resale is possible, bidder 1 can implement this optimal mechanism by reselling at the price $wH(\rho(w, \gamma(b))) + \rho(w, \gamma(b))(H(\gamma(b)) - H(\rho(w, \gamma(b))))$ to the highest-value bidder (with value w) among bidders 2 to $n-1$, who will then offer to resell only to bidder n at the price $\rho(w, \gamma(b))$.

[67] When further resale is possible and the resale mechanism is implemented as in the previous footnote, the second term in the factor between braces is $\int_v^{\alpha(b)} (wH(\rho(w, \gamma(b))) + \rho(w, \gamma(b))(H(\gamma(b)) - H(\rho(w, \gamma(b)))) - b)\, dF(w)^{n-2}$ instead. The same FOC results.

If bidder n wins with such a bid, there is no profitable resale, since $\alpha(b) \leq r(b) < v$. If bidder n raises his bid from b to db, the only first-order effect, beyond the effect on his payment at auction, occurs when the highest bid from bidders 1 to $n-1$ is b, that is, when there highest value is $\alpha(b)$. In this case, by rasing his bid and winning the auction, bidder n saves the price $\rho(\alpha(b), \gamma(b))$, he would accept at resale, and the first term above follows.

We then obtain the FOC below:

$$(n-1)(r(b)-b)\frac{d}{db}\ln F(\alpha(b)) = 1.\text{(A8.6)}$$

Notice that when this FOC is satisfied, bidder n is indifferent between all bids in $[c, r^{-1}(v)]$. The derivative of bidder n's expected payoff at b in $(r^{-1}(v), \beta(v))$ is:

$$(v-b)\frac{d}{db}F(\alpha(b))^{n-1} - F(\alpha(b))^{n-1};\text{(A8.7)}$$

and at $b > \beta(v)$:

$$(\alpha(b)-b)\frac{d}{db}F(\alpha(b))^{n-1} - F(\alpha(b))^{n-1}.\text{(A8.8)}$$

System of FOC's

The system of FOC's can be written as (19, 20). Because the derivatives (A8.1), (A8.3), (A8.5), (A8.7), (A8.8) above, of the payoffs with respect to the bid, are nondecreasing in the value, a standard argument shows that the system (19, 20), along with the boundary conditions $\beta(c) = \delta(c) = c, \beta(d) = \delta(d)$ and the strict monotonicity of α, δ, is a sufficient condition for an equilibrium.

The FD Regime

Consider a PBE with bidding functions β, δ of the FPA under PD as in Theorem 4 (i). Under FD, apply the randomization procedure to bidder n's bidding function δ and keep his resale strategy unchanged. Assume that every bidder $i = 1, ..., n-1$ follows the same bidding function β and the resale strategy described in the main text. These transformed strategies then form an equivalent PBE.

In fact, the derivative of bidder 1's expected payoff at $b < \beta(v)$ is equal to (A8.9)

$$\begin{pmatrix} vH(r(\beta(v))|b) + \\ r(\beta(v))[1 - H(r(\beta(v))|b)] - b \end{pmatrix} F(\alpha(b))^{n-1} \frac{d}{db} H(\gamma(b))$$
$$+ \int_c^b \left\{ \begin{array}{c} vH(r(\beta(v))|b') + \\ r(\beta(v))[1 - H(r(\beta(v))|b')] - b \end{array} \right\} dH(\gamma(b')) \frac{d}{db} F(\alpha(b))^{n-1}$$
$$-H(\gamma(b)) F(\alpha(b))^{n-1}. \text{ (A8.9)}$$

Bidder 1's optimal resale mechanism after winning with a bid b, even after observing bidder n's losing bid, is to offer bidder n the same resale price $r(\beta(v))$. The first and second term follow (by the randomization procedure, the marginal distribution of bidder n's bid is the same as under PD). Since it accounts for the change of the payment at auction when winning, the final term is obviously the same as under PD.

Because, from the randomization procedure, $r(b)$ is the minimum of the support of $H(.|b)$ and bidder n's marginal bid distribution and his probability, over all bids smaller than b, of accepting $r(b)$ are the same as under partial disclosure, (A8.9) reduces to the LHS of (A8.1) when $v = \alpha(b)$.

The derivative of bidder 1's expected payoff at $b > \beta(v)$ is equal to

(A8.10) below:

$$\begin{aligned}&(r(b)-b)\,F(\alpha(b))^{n-1}\frac{d}{db}H(\gamma(b))\\&+\left\{\begin{array}{c}(\alpha(b)-b)\,H(r(b))+\\(r(b)-b)\,[H(\gamma(b))-H(r(b))]\end{array}\right\}\frac{d}{db}F(\alpha(b))^{n-1}\\&-H(\gamma(b))\,F(\alpha(b))^{n-1}.\ \text{(A8.10)}\end{aligned}$$

Conditional on bidder n's submitting b, his virtual value is larger than $\alpha(b)$ with probability one, since, by the randomization procedure, the virtual value at the minium $r(b)$ of the support is $\alpha(b)$. After observing the bid b from bidder n and smaller bids from the other bidders, it is then optimal to make a resale offer only to bidder n. Moreover, the optimal resale price is the minimum $r(b)$ of the support, since it would be optimal if bidder 1's value was $\alpha(b) > v$. The first term follows.

If the highest bid from bidders 2 to $n-1$ is b, that is, if their highest value is $\alpha(b)$, and if bidder n has submitted a smaller bid, the optimal resale mechanism consists, as under PD, in selling to bidder n if his virtual value is larger than $\alpha(b)$, that is, by the randomization procedure, if his value is larger than $r(b)$, and to the highest bidder, among bidders 2 to n, otherwise. By the randomization procedure again, the marginal probabilities that bidder n accepts the resale price are the same as under PD and the second term above follows. The reason for the third term is as in the first case above.

The derivative (A8.10) is equal to the LHS of (A8.2) and we obtain the same FOC (A8.4) for bidder 1 as under PD. Since bidders 1 to n make the same bids and resale offers, when they follow their strategies, the derivative of bidder n's expected payoffs and the FOC for this bidder are obviously the same.

Because the derivatives above are nondecreasing functions of the value v, the FOC's, which, by assumption, β and δ satisfy, are sufficient and the randomization procedure produces a PBE.

Appendix 9
Technical Extension of the Function ρ

For $v \leq w$, the function $\rho(v,w)$ is defined according to (1), that is:

$$v = \rho(v,w) - \frac{H(w) - H(\rho(v,w))}{h(\rho(v,w))}.\text{(A9.1)}$$

For technical purposes, extend the function $\rho(v,w)$ to $[c,w]^2$ by setting $\rho(v,w) = \frac{v+w}{2}$, for $v > w$. It is then easy to check that the so defined ρ is continuously differentiable over $(c,d]^2$. In fact, from Assumption (ii), (A9.1) satisfies the conditions of the implicit function theorem (the derivative with respect to ρ of the RHS of (A9.1) is strictly positive). Furthermore, the partial derivatives of the solution ρ of (A9.1) tend towards 1/2 when (v,w) tend towards towards a couple on the 45-degree line. Since ρ is continuously differentiable, it is also locally Lipschitz over $(c,d]^2$.

Lemma A9.1: *$\frac{\partial}{\partial v}\rho(v,w)$ is bounded away from zero over $[c+\varepsilon,d]^2$, for all $\varepsilon > 0$.*

Proof: For all (v,w) with $v \geq w$, we have, by definition of the extension of ρ, $\frac{\partial}{\partial v}\rho(v,w) = \frac{1}{2}$. From (A9.1), we have, for $v \leq w$:

$$\left\{2 + \frac{H(w) - H(\rho(v,w))}{h(\rho(v,w))^2}h'(\rho(v,w))\right\}\frac{\partial}{\partial v}\rho(v,w) = 1.$$

If, furthermore, $(v,w) \in [c+\varepsilon,d]^2$, then $\rho(v,w) \in [c+\varepsilon,d]^2$. From the equality above, we have:

$$\frac{\partial}{\partial v}\rho(v,w) \geq L(\varepsilon) > 0,$$

for all (v,w) in $[c+\varepsilon,d]^2$such that $v \leq w$, with $L(\varepsilon)$ defined as follows:

$$L(\varepsilon) = \frac{1}{2 + K(\varepsilon)/M(\varepsilon)},$$

with:

$$\begin{aligned} K(\varepsilon) &= \max_{v \in [c+\varepsilon, d]} h'(v) \\ M(\varepsilon) &= \min_{v \in [c+\varepsilon, d]} h(v) > 0. \end{aligned}$$

||

Main Lemmas

Through the change of variables $\psi_1 = F\alpha$, $\psi_{n+1} = H\gamma$, the system (19, 20) becomes the system below:

$$\begin{aligned} & \frac{d}{db}\psi_n(b) \\ = \;& \frac{\psi_n(b)}{\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_1(b)\right) - b} \\ & \left\{ 1 - \frac{n-2}{n-1} \frac{\begin{array}{c} \left(F^{-1}\psi_1(b) - b\right) H\left(\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right)\right) + \\ \left(\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right) - b\right) \\ \left(\psi_n(b) - H\left(\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right)\right)\right) \end{array}}{\left(\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right) - b\right)\psi_n(b)} \right\}; \\ & \frac{d}{db}\psi_1(b) \\ = \;& \frac{\psi_1(b)}{(n-1)\left(\rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right) - b\right)}. \end{aligned}$$

By extending the functions F^{-1}, H^{-1} into locally Lipschitz functions over $(0, 1+\varepsilon)$, where $\varepsilon > 0$, in such a way that $\frac{1-q}{h(H^{-1}(q))}$ is nonincreasing over this interval, the assumptions of the theory of ordinary differential equations are satisfied over the domain[68]. $\mathcal{D} = \{(b, \psi_1, \psi_n) \,|\, 0 < \psi_1, \psi_n \leq 1, \rho\left(F^{-1}\psi_1(b), H^{-1}\psi_n(b)\right) > b\}$.

[68] The change of variables allows to apply the theory of ordinary differential equations without making unnecessary Lipschitz assumptions on the density f.

Consequently, for every $\eta < d$, there exists one and one solution in this domain

Consider next the initial system (19, 20) over the domain $D = \left\{ \begin{array}{c} (b, \alpha, \gamma) \mid \\ c < \alpha, \gamma \leq d; \rho(\alpha, \gamma) > b \end{array} \right\}$ (the image of $\mathcal{D}$ by the change of variables above), with initial condition (A9.4) below, where η is a parameter such that $\eta < d$:

$$\begin{aligned} & \frac{d}{db} \ln H(\gamma(b)) \\ = & \frac{1}{\rho(\alpha(b), \gamma(b)) - b} \\ & \left\{ 1 - \frac{n-2}{n-1} \frac{\begin{array}{c} (\alpha(b) - b) H(\rho(\alpha(b), \gamma(b))) + \\ (\rho(\alpha(b), \gamma(b)) - b) \\ (H(\gamma(b)) - H(\rho(\alpha(b), \gamma(b)))) \end{array}}{(\rho(\alpha(b), \gamma(b)) - b) H(\gamma(b))} \right\}; \text{(A9.2)} \\ & \frac{d}{db} \ln F(\alpha(b)) \\ = & \frac{1}{(n-1)(\rho(\alpha(b), \gamma(b)) - b)} . \text{(A9.3)} \end{aligned}$$

$$\alpha(\eta) = \gamma(\eta) = d. \text{(A9.4)}$$

It follows immediately from (A9.3) that $\frac{d}{db} \alpha(b) > 0$, for all solution of (19,20) in D. Moreover, at the initial condition, the derivative of γ is also stricltyre strictly positive, in fact, from (19,20) and $\rho(d, d) = d$:

$$\frac{d}{db} \ln H(\gamma(b)) = \frac{d}{db} \ln F(\alpha(b)) = \frac{1}{(n-1)(c-d)}.$$

We have proved Lemma A9.2 below.

Lemma A9.2: *Let (α, γ) be a solution of (A9.2-A9.4) in the domain D*

defined over $(b', \eta]$. *Then,* $\frac{d}{db}\alpha(b) > 0$*, for all* b *in* $(b', \eta]$*, and* $\frac{d}{db}\gamma(\eta) > 0$*.*

From Lemma A9.2, the solution to (A9.2-A9.4) is strictly increasing at η and can be continued within D to the left of this point.

Lemma A9.3: *Let* (α, γ) *be a solution of (A9.2-A9.4) in the domain* D *defined over* $(b', \eta]$*. Since, from Lemma A9.2,* $\frac{d}{db}\alpha(b) > 0$*, for all* b *in* $(b', \eta]$*, the function* φ *below is well defined and differentiable:*

$$\varphi = \gamma\alpha^{-1} = \gamma\beta,$$

where β *is the inverse of* α*. Then, the inequality below holds true for all* v *in* $(\alpha(b'), d]$*:*

$$\lambda(v) \leq \varphi(v);$$

where λ *is defined as follows:*

$$\lambda(v) = H^{-1}\left(F(v) \min_{w \in [v,d]} \frac{H(w)}{F(w)}\right).$$

Proof: Let v be in $(\alpha(b'), d)$, k be such that $0 < k < \min_{w \in [v,d]} \frac{H(w)}{F(w)}$, and let the function λ_k be defined as follows:

$$\lambda_k(w) = H^{-1}(kF(w)),$$

for all w in $(v, d]$. From its definition and $k < \frac{H(w)}{F(w)}$, for all $w \geq v$, we have:

$$\begin{aligned} \frac{d}{dv} \ln H(\lambda_k(w)) &= \frac{d}{dv} \ln F(w) \\ \lambda_k(w) &< w, \end{aligned}$$

for all $w \geq v$. In particular, $\lambda_k(d) < d$ and, thus,

$$\lambda_k(d) < \varphi(d).$$

(3) and (4) can be rewritten as follows:

$$\begin{aligned}
&\left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)\frac{d}{dv}\ln H\left(\varphi\left(v\right)\right)+\\
&\left(n-2\right)\left\{\begin{array}{c}\left(v-\beta\left(v\right)\right)\frac{H\left(\rho\left(v,\varphi\left(v\right)\right)\right)}{H\left(\varphi\left(v\right)\right)}+\\ \left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)\left[1-\frac{H\left(\rho\left(v,\varphi\left(v\right)\right)\right)}{H\left(\varphi\left(v\right)\right)}\right]\end{array}\right\}\frac{d}{dv}\ln F\left(v\right)\\
=\;&\frac{d}{dv}\beta\left(v\right), \text{ (A9.5)}
\end{aligned}$$

$$\begin{aligned}
&\left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)\frac{d}{dv}\ln F\left(v\right)+\\
&\left(n-2\right)\left\{\begin{array}{c}\left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)\frac{H\left(\rho\left(v,\varphi\left(v\right)\right)\right)}{H\left(\varphi\left(v\right)\right)}+\\ \left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)\left[1-\frac{H\left(\rho\left(v,\varphi\left(v\right)\right)\right)}{H\left(\varphi\left(v\right)\right)}\right]\end{array}\right\}\frac{d}{dv}\ln F\left(v\right)\\
=\;&\frac{d}{dv}\beta\left(v\right). \text{(A9.6)}
\end{aligned}$$

Suppose there exists u in $(v, d]$ such that $\varphi(u) = \lambda_k(u)$. Since $\lambda_k(u) < u$, we have $\varphi(u) < u$ and, consequently, $\rho(\varphi(u), u) < u$ (because $\min(v, w) < \rho(v, w) < \max(v, w)$, for all (v, w) such that $v \neq w$). From (A9.5) and (A9.6), we then have $\frac{d}{dv}\ln H(\varphi(u)) < \frac{d}{dv}\ln F(u)$. Since $\frac{d}{dv}\ln H(\lambda_k(u)) = \frac{d}{dv}\ln F(u)$, we then have $\frac{d}{dv}\ln H(\varphi(u)) < \frac{d}{dv}\ln H(\lambda_k(u))$.

From an (elementary) technical lemma, we obtain $\varphi(v) \geq H^{-1}(kF(v))$. The result then follows by taking the limit for k tending towards $\min_{w\in[v,d]} \frac{H(w)}{F(w)}$.
||

Lemma A9.4: *Let (α, γ) be a solution of (A9.2-A9.4) in D defined over $(b', \eta]$. Then, the inequality below holds true for all v in $(\alpha(b'), d]$:*

$$v \leq \varphi(v),$$

and

$$\alpha(b) \leq \gamma(b),$$

for all b in $(b', \eta]$.

Proof: It suffices to apply the previous lemma and to notice that, under our assumption of stochastic dominance (iii), $\min_{w\in[v,d]} \frac{H(w)}{F(w)} = \frac{H(v)}{F(v)}$. ||

Lemma A9.5: *Let (α, γ) be a solution of (A9.2-A9.4) in D defined over $(b', \eta]$. Then, $\frac{d}{db}\alpha(b), \frac{d}{db}\gamma(b) > 0$, $\frac{d}{db}\frac{H(\gamma(b))}{F(\alpha(b))} \geq 0$ and $H(\gamma(b)) \leq F(\alpha(b))$, for all b in $(b', \eta]$.*

Proof: From Lemma A9.2, $\frac{d}{db}\alpha(b) > 0$, for all b in $(b', \eta]$. From Lemma A9.4, $\alpha(b) \leq \gamma(b)$. Consequently, the sum of $(\alpha(b) - b)\frac{H(\rho(\alpha(b),\gamma(b)))}{H(\gamma(b))}$ and $(\rho(\alpha(b), \gamma(b)) - b)\left[1 - \frac{H(\rho(\alpha(b),\gamma(b)))}{H(\gamma(b))}\right]$ is not smaller than $\rho(\alpha(b), \gamma(b)) - b$, and the factor between braces in the RHS of (A9.2) is not smaller than $1 - \frac{n-2}{n-1} > \frac{1}{n-1} > 0$. We then also have $\frac{d}{db}\gamma(b) > 0$, for all b in $(b', \eta]$. Moreover, from (A9.2) and (A9.3), we find $\frac{d}{db}\ln H(\gamma(b)) \geq \frac{1}{\rho(\alpha(b),\gamma(b))-b}\frac{1}{n-1} = \frac{d}{db}\ln F(\alpha(b))$, for all b in $(b', \eta]$, and consequently $\frac{H(\gamma(b))}{F(\alpha(b))}$ is nondecreasing over this interval. Since, from (A9.4), $\frac{H(\gamma(\eta))}{F(\alpha(\eta))} = 1$, we obtain $H(\gamma(b)) \leq F(\alpha(b))$, for all b in $(b', \eta]$. ||

Lemma A9.6 (Monotonicity of the solution of (A9.2-A9.4) with respect to η): *Let (α, γ) and $(\widetilde{\alpha}, \widetilde{\gamma})$ be the solutions of (A9.2, A9.3) in D and the initial condition (A9.4) for η and $\widetilde{\eta}$, respectively, with $\widetilde{\eta} < \eta$. Assume further that (α, γ) and $(\widetilde{\alpha}, \widetilde{\gamma})$ are defined over $(\underline{b}, \widetilde{\eta}]$. Then, we have:*

$$\begin{aligned} \widetilde{\alpha}(b) &> \alpha(b) \\ \widetilde{\gamma}(b) &> \gamma(b), \end{aligned}$$

for all b in $(\underline{b}, \widetilde{\eta}]$.

Proof: There exists no b in $(\underline{b}, \widetilde{\eta}]$ such that $\widetilde{\alpha}(b) = \alpha(b)$ and $\widetilde{\gamma}(b) = \gamma(b)$. Otherwise, (α, γ) and $(\widetilde{\alpha}, \widetilde{\gamma})$ could be extended over the unions of their definition domains and would coincide over this union. However, this is impossible since, from (A9.4) and Lemma A9.2 :

$$\widetilde{\alpha}\left(\widetilde{\eta}\right) = d > \alpha\left(\widetilde{\eta}\right).$$

Let b' be defined as follows:

$$b' = \inf\left\{b \in \left[\underline{b}, \widetilde{\eta}\right] \left|\widetilde{\alpha}\left(b''\right) > \alpha\left(b''\right), \widetilde{\gamma}\left(b''\right) > \gamma\left(b''\right), \text{ for all } b'' \text{ in } \left(b, \widetilde{\eta}\right]\right.\right\}.$$

From our assumptions and by continuity, there exists $\varepsilon > 0$ such that $\left[\widetilde{\eta} - \varepsilon, \widetilde{\eta}\right]$ is included in the set in the definition above of b'. We want to prove that $b' = \underline{b}$. Suppose $b' > \underline{b}$. Then, by continuity and from the observation above only the two cases below are possible:

Case 1: $\widetilde{\alpha}\left(b'\right) = \alpha\left(b'\right)$ and $\widetilde{\gamma}\left(b'\right) > \gamma\left(b'\right)$.
Case 2: $\widetilde{\alpha}\left(b'\right) > \alpha\left(b'\right)$ and $\widetilde{\gamma}\left(b'\right) = \gamma\left(b'\right)$.

We investigate each case in turn.

Case 1. From (A9.3) and because ρ is strictly increasing, we have:

$$\begin{aligned} & \frac{d}{db} \ln F\left('\alpha\left(b\right)\right) \\ = \; & \frac{1}{\left(n-1\right)\left(\rho\left(\alpha\left(b'\right), \gamma\left(b'\right)\right) - b'\right)} \\ > \; & \frac{1}{\left(n-1\right)\left(\rho\left(\widetilde{\alpha}\left(b'\right), \widetilde{\gamma}\left(b'\right)\right) - b'\right)} \\ = \; & \frac{d}{db} \ln F\left(\widetilde{\alpha}\left(b'\right)\right). \end{aligned}$$

Then, since $\widetilde{\alpha}\left(b'\right) = \alpha\left(b'\right)$ and $\frac{d}{db}\widetilde{\alpha}\left(b'\right) < \frac{d}{db}\alpha\left(b'\right)$, we would have $\widetilde{\alpha}\left(b\right) < \alpha\left(b\right)$, for some b to the right of b', which would contradict the definition of b'.

Case 2. (A9.2) can be rewritten as follows:

$$\begin{aligned}
& \left(\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)-b\right)\frac{d}{db}\ln H\left(\gamma\left(b\right)\right)+ \\
& \frac{n-2}{n-1}\left\{1-\frac{\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)-\alpha\left(b\right)}{\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)-b}\frac{H\left(\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)\right)}{H\left(\gamma\left(b\right)\right)}\right\} \\
= & \ 1. \ \text{(A9.7)}
\end{aligned}$$

From Lemma A9.4, $\alpha\left(b\right)\leq\gamma\left(b\right)$. From the definition (A9.1) of ρ, $\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)-\alpha\left(b\right)$ is equal to $\frac{H(\gamma(b))-H(\rho(\alpha(b),\gamma(b)))}{h(\rho(\alpha(b),\gamma(b)))}$ and the term between braces in the LHS of (A9.7) is equal to:

$$1-\frac{H\left(\gamma\left(b\right)\right)-H\left(\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)\right)}{H\left(\gamma\left(b\right)\right)}\frac{H\left(\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)\right)}{h\left(\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)\right)},$$

which, from our assumption (ii) is nondecreasing with respect $\alpha\left(b\right)$. Because $\rho\left(\alpha\left(b\right),\gamma\left(b\right)\right)$ is strictly increasing in $\alpha\left(b\right)$, we find, under the assumptions of Case 2: $\frac{d}{db}\ln H\left(\widetilde{\gamma}\left(b'\right)\right)<\frac{d}{db}\ln H\left(\gamma\left(b'\right)\right)$, which, together with $\widetilde{\gamma}\left(b'\right)=\gamma\left(b\right)$, contradicts the definition of b'. ||

Lemma A9.7: *Let $\underline{b}\left(\eta\right)$ be the lower-extremity of the maximal definition interval of the solution $\left(\alpha,\gamma\right)$ in D of the system (A9.2, A9.3) with initial condition (A9.4) for the value η of the parameter. Then, $\underline{b}\left(\eta\right)$ is strictly increasing when strictly above c. Furthermore, if $\underline{b}\left(\eta\right)>c$, then $\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)>c$, $\rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right)=\underline{b}\left(\eta\right)$.*

Proof: Let η be such that $\eta<d$ and $\underline{b}\left(\eta\right)>c$. From the definition of D and Lemma A9.5, we have $\alpha\left(\underline{b}\left(\eta\right)\right)=c,\gamma\left(\underline{b}\left(\eta\right)\right)=c$, or $\rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right)=\underline{b}\left(\eta\right)$, where $\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)$ are the values of the continuous extensions of the solution $\left(\alpha,\gamma\right)$ to the system (A9.2, A9.3) with initial condition (A9.4) with the value η of the parameter. From Lemmas A9.4 and A9.5, $\alpha\left(\underline{b}\left(\eta\right)\right)=c$ if and only if $\gamma\left(\underline{b}\left(\eta\right)\right)=c$, in which case $\rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right)=c<\underline{b}\left(\eta\right)$, contrary to the definition of D. Consequently, we have

$$\begin{aligned}\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right) &> c \text{ (A9.8)}\\ \rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right) &= \underline{b}\left(\eta\right).\end{aligned}$$

Let $\widetilde{\eta}$ be such that $\eta < \widetilde{\eta} < d$. Let $(\widetilde{\alpha}, \widetilde{\gamma})$ be the solution of the (A9.2-A9.4) for the values $\widetilde{\eta}$ of the parameter in the initial condition (A9.4). The inequality $\underline{b}\left(\widetilde{\eta}\right) < \underline{b}\left(\eta\right)$ is impossible. In fact, from Lemma A9.6, we have:

$$\begin{aligned}\gamma\left(b\right) &> \widetilde{\gamma}\left(b\right), \text{(A9.9)}\\ \alpha\left(b\right) &> \widetilde{\alpha}\left(b\right), \text{(A9.10)}\end{aligned}$$

for all b in $\left(\max\left(\underline{b}\left(\widetilde{\eta}\right),\underline{b}\left(\eta\right)\right),\eta\right)$. Suppose $\underline{b}\left(\widetilde{\eta}\right) < \underline{b}\left(\eta\right)$. By making b in these inequalities tend towards $\underline{b}\left(\eta\right)$, we would obtain $\widetilde{\gamma}\left(\underline{b}\left(\eta\right)\right) \leq \gamma\left(\underline{b}\left(\eta\right)\right)$ and $\widetilde{\alpha}\left(\underline{b}\left(\eta\right)\right) \leq \alpha\left(\underline{b}\left(\eta\right)\right)$ and, consequently, $\underline{b}\left(\eta\right) = \rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right) \geq \rho\left(\alpha\left(\underline{b}\left(\widetilde{\eta}\right)\right),\gamma\left(\underline{b}\left(\widetilde{\eta}\right)\right)\right) = \underline{b}\left(\widetilde{\eta}\right)$, a contradiction.

We have proved $\underline{b}\left(\eta\right) \leq \underline{b}\left(\widetilde{\eta}\right)$. We now prove $\underline{b}\left(\eta\right) < \underline{b}\left(\widetilde{\eta}\right)$ by showing that the equality $\underline{b}\left(\eta\right) = \underline{b}\left(\widetilde{\eta}\right)$ is impossible. Suppose $\underline{b}\left(\eta\right) = \underline{b}\left(\widetilde{\eta}\right)$. We then have, from the definition of D and from (A9.9) and (A9.10):

$$\begin{aligned}\widetilde{\gamma}\left(\underline{b}\left(\eta\right)\right) &\leq \gamma\left(\underline{b}\left(\eta\right)\right)\\ \widetilde{\alpha}\left(\underline{b}\left(\eta\right)\right) &\leq \alpha\left(\underline{b}\left(\eta\right)\right)\\ \rho\left(\widetilde{\alpha}\left(\underline{b}\left(\eta\right)\right),\widetilde{\gamma}\left(\underline{b}\left(\eta\right)\right)\right) &= \rho\left(\alpha\left(\underline{b}\left(\eta\right)\right),\gamma\left(\underline{b}\left(\eta\right)\right)\right) = \underline{b}\left(\eta\right).\end{aligned}$$

Since ρ is strictly increasing, we find:

$$\begin{aligned}\widetilde{\gamma}\left(\underline{b}\left(\eta\right)\right) &= \gamma\left(\underline{b}\left(\eta\right)\right)\\ \widetilde{\alpha}\left(\underline{b}\left(\eta\right)\right) &= \alpha\left(\underline{b}\left(\eta\right)\right). \text{ (A9.11)}\end{aligned}$$

From (A9.3) and (A9.9,A9.10), we have:

$$
\begin{aligned}
& \frac{d}{db}\ln F(\alpha(b)) \\
= & \frac{1}{(n-1)(\rho(\alpha(b),\gamma(b))-b)} \\
< & \frac{1}{(n-1)(\rho(\widetilde{\alpha}(b),\widetilde{\gamma}(b))-b)} \\
= & \frac{d}{db}\ln F(\widetilde{\alpha}(b)),
\end{aligned}
$$

for all b in $(\underline{b}(\eta),\eta]$, and, consequently, $\frac{F(\alpha(b))}{F(\widetilde{\alpha}(b))}$ is strictly decreasing over this interval.

From (A9.8) and (A9.11), we have $\frac{F(\alpha(\underline{b}(\eta)))}{F(\widetilde{\alpha}(\underline{b}(\eta)))}=1$. Thus, $\alpha(b)<\widetilde{\alpha}(b)$, for all b in $(\underline{b}(\eta),\eta]$, which contradicts Lemma A9.6. ||

In what follows, $\underline{b}(\eta)$ is as defined in Lemma A9.7. The sub-lemma below is helpful in the proof of Lemma A9.8.

Sub-lemma A9.1: *For all v in $(\alpha(\underline{b}),d]$, with $\varphi=\gamma\beta$ and $\underline{b}\geq\underline{b}(\eta)$:*

$$
\begin{aligned}
& (\rho(v,\varphi(v))-\beta(v))F(v)^{n-1}-(\rho(\alpha(\underline{b}),\gamma(\underline{b}))-\underline{b})F(\alpha(\underline{b}))^{n-1} \\
= & \int_{\alpha(\underline{b})}^{v} F(v)^{n-1}\frac{d}{dv}\rho(v,\varphi(v))\,;\text{(A9.12)}
\end{aligned}
$$

$$
\begin{aligned}
& \beta(v)F(v)^{n}-\underline{b}F(\alpha(\underline{b}))^{n} \\
= & \int_{\alpha(\underline{b})}^{v}\rho(v,\varphi(v))\frac{d}{dv}F(v)^{n}.\ \text{(A9.13)}
\end{aligned}
$$

Proof: From (A9.3), we have:

$$
(\rho(v,\varphi(v))-\beta(v))\frac{d}{dv}F(v)^{n-1}=F(v)^{n-1}\frac{d}{dv}\beta(v),
$$

and hence:

$$\frac{d}{dv}\left\{\left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)F\left(v\right)^{n-1}\right\}=F\left(v\right)^{n-1}\frac{d}{dv}\rho\left(v,\varphi\left(v\right)\right),$$

for all v in $\left(\alpha\left(\underline{b}\right),d\right]$, with $\varphi=\gamma\beta$ and $\underline{b}\geq\underline{b}\left(\eta\right)$. Integrating this equation from $\alpha\left(\underline{b}\right)$ to v in $\left(\alpha\left(\underline{b}\right),d\right]$, we find:

$$\begin{aligned}&\left(\rho\left(v,\varphi\left(v\right)\right)-\beta\left(v\right)\right)F\left(v\right)^{n-1}-\left(\rho\left(\alpha\left(\underline{b}\right),\gamma\left(\underline{b}\right)\right)-\underline{b}\right)F\left(\alpha\left(\underline{b}\right)\right)^{n-1}\\=&\int_{\alpha\left(\underline{b}\right)}^{v}F\left(v\right)^{n-1}\frac{d}{dv}\rho\left(v,\varphi\left(v\right)\right).\end{aligned}$$

(A9.12) then follows. Integrating (A9.12) by parts, we find (A9.13). ||

Lemma A9.8:
(i) For all $\eta\leq c$, we have $\underline{b}\left(\eta\right)\leq c$;
(ii) For all η in $\left(d-\int_{c}^{d}H\left(w\right)^{n-1}dw,d\right)$, we have $\underline{b}\left(\eta\right)>c$.

Proof: **(i)** If $\eta\leq c$, then, since $\left(\eta,d,d\right)$ belongs D, Lemma A9.5 implies that there exists a strictly increasing solution of (A9.2-A9.4) that can be continued strictly to the left of η. Consequently, $\underline{b}\left(\eta\right)<c$ and (i) is proved.

(ii) Let η be in the open interval $\left(d-\int_{c}^{d}H\left(w\right)^{n-1}dw,d\right)$. We show that $\underline{b}\left(\eta\right)>c$. Suppose that $\underline{b}\left(\eta\right)\leq c$, instead. From (A9.13) in Sub-lemma A9.1 with $v=d$ and $\underline{b}=c$ and Lemma A9.5, which implies $\varphi=\gamma\beta\leq H^{-1}F$, we have:

$$
\begin{aligned}
& \eta \\
\leq & \int_{\alpha(\underline{b})}^{d} \rho(v, \varphi(v)) \frac{d}{dv} F(v)^{n-1} \\
\leq & \int_{\alpha(\underline{b})}^{d} H^{-1} F(v) \frac{d}{dv} F(v)^{n-1} \\
\leq & \int_{c}^{d} H^{-1} F(v) \frac{d}{dv} F(v)^{n-1} \\
= & \; d - \int_{c}^{d} H(w)^{n-1} dw,
\end{aligned}
$$

and $\eta \leq d - \int_c^d H(w)^{n-1} dw$, which contradicts our initial assumption. ||

Let η^* be defined as follows:

$$\eta^* = \inf \{\eta < d | \underline{b}(\eta) \geq c\}.$$

From Lemma A9.8 (ii), the set in the definition of η^* is not empty and:

$$c \leq \eta^* \leq d - \int_c^d H(w)^{n-1} dw. \text{(A9.14)}$$

Lemma A9.9: *Let $(\alpha(b;\eta), \gamma(b;\eta))$ be the solution of (A9.2-A9.4) in the domain D. Suppose $\underline{b}(\eta^*) > c$. Then, $\underline{b}(\eta) < c$, for all $\eta < \eta^*$, and :*

$$
\begin{aligned}
& \lim_{\eta \to_< \eta^*} \rho(\alpha(b;\eta), \gamma(b;\eta)) \\
= & \lim_{\eta \to_< \eta^*} \alpha(b;\eta) \\
= & \lim_{\eta \to_< \eta^*} \gamma(b;\eta) \\
= & \; \underline{b}(\eta^*),
\end{aligned}
$$

for all $b \in (c, \underline{b}(\eta^*))$.

Proof: For all $\eta \leq \eta^*$, we have $(c, \underline{b}(\eta^*))$ is included the (interior) of the definition domain of the solution (α, γ) of (A9.2-A9.4) and

$$\begin{aligned} \alpha(b) &\leq \rho(\alpha(b), \gamma(b)) \\ &\leq \rho(\alpha(\underline{b}(\eta^*)), \gamma(\underline{b}(\eta^*))) \\ &\leq \rho(\alpha^*(\underline{b}(\eta^*)), \gamma^*(\underline{b}(\eta^*))) \\ &= \underline{b}(\eta^*), \text{ (A9.15)} \end{aligned}$$

for all b in this interval, where the first inequality follows from Lemma A9.4, the second from Lemma A9.5, the third from Lemma A9.6, and the the equality from Lemmas A9.7, A9.4, and A9.5.

Let b be in $(c, \underline{b}(\eta^*))$. Suppose $\lim_{\eta \to_< \eta^*} \alpha(b)$ does not exist or is different from $\underline{b}(\eta^*)$. Then, there exists $v' < \underline{b}(\eta^*)$ and a sequence $(\eta_k)_{k \geq 1}$ such that

$$\begin{aligned} \lim_{k \to +\infty} \eta_k &= \eta^* \\ \alpha(b; \eta_k) &< v' \\ \eta_k &< \eta^*, \end{aligned}$$

for all $k \geq 1$.

Let ε be a strictly positive number. Let (α, γ) be a solution defined over $(\underline{b}, \eta]$ of (A9.2-A9.4). Then, (β, φ) is the solution, defined over $(\alpha(\underline{b}), d]$, of the system below:

$$\frac{d}{dv}\varphi(v) = \frac{f(v)}{F(v)}\frac{H(\varphi(v))}{h(\varphi(v))} \left\{ n-1-(n-2)\frac{\begin{array}{c}(v-\beta(v))H(\rho(v,\varphi(v)))+\\(\rho(v,\varphi(v))-\beta(v))\\(H(\varphi(v))-H(\rho(v,\varphi(v))))\end{array}}{(\rho(v,\varphi(v))-\beta(v))H(\varphi(v))} \right\} \text{(A9.16)}$$

$$\frac{d}{dv}\beta(v) = (n-1)\frac{f(v)}{F(v)}(\rho(v,\varphi(v))-\beta(v)) . \text{(A9.17)}$$

Through the change of variables $(p,\chi,\zeta) = (F(v), H\varphi F^{-1}, \beta F^{-1})$, the system above is equivalent to the system below:

$$\frac{d}{dp}\chi(p) = \frac{\chi(p)}{p} \left\{ n-1-(n-2)\frac{\begin{array}{c}(F^{-1}(p)-\zeta(p))H(\rho(F^{-1}(p),\chi(p)))+\\(\rho(F^{-1}(p),\chi(p))-\rho(p))\\(\chi(p)-H(\rho(F^{-1}(p),\chi(p))))\end{array}}{(\rho(F^{-1}(p),\chi(p))-\rho(p))\chi(p)} \right\} \text{(A9.18)}$$

$$\frac{d}{dp}\rho(p) = \frac{n-1}{p}\left(\rho\left(F^{-1}(p),\chi(p)\right)-\rho(p)\right) \text{ (A9.19)},$$

and $(H\varphi F^{-1}, \beta F^{-1})$ is a solution over $(F(\alpha(\underline{b})), 1]$ to thissyet and the initial condition below:

$$\chi(1) = 1, \rho(1) = \eta.$$

For the sake of convenience, denote (α^*, γ^*) the solution to (A9.2-A9.4) for η^*. Let w be strictly larger than $\alpha^*\left(\underline{b}\left(\eta^*\right)\right)$ such that:

$$\left|\rho\left(w, \varphi^*\left(w\right)\right) - \beta^*\left(w\right)\right| < \varepsilon$$

(such a w exists since, from Lemmas A9.7, A9.4, and A9.5, $\rho\left(\alpha^*\left(\underline{b}\left(\eta^*\right)\right), \varphi^*\left(\alpha^*\left(\underline{b}\left(\eta^*\right)\right)\right)\right) = \beta^*\left(\alpha^*\left(\underline{b}\left(\eta^*\right)\right)\right)$). From the continuity of the solution to the system with initial condition above and the continuity of H^{-1} and ρ, for all $\varepsilon' > 0$, there exists $\delta > 0$, such that ζ and χ is defined at $F\left(w\right)$, and thus β and $\varphi = H^{-1}\chi F$ are defined at w, and such that:

$$\begin{aligned} & \left|\zeta\left(F\left(w\right)\right) - \zeta^*\left(F\left(w\right)\right)\right| \\ = & \left|\beta\left(w\right) - \beta^*\left(w\right)\right| \\ < & \varepsilon, \end{aligned}$$

$$\begin{aligned} & \left|\rho\left(w, \varphi\left(w\right)\right) - \rho\left(w, \varphi^*\left(w\right)\right)\right| \\ = & \left|\rho\left(w, H^{-1}\chi F\left(w\right)\right) - \rho\left(w, H^{-1}\chi^* F\left(w\right)\right)\right| \\ < & \varepsilon, \end{aligned}$$

for all η such that $\eta^* - \delta < \eta < \eta^*$ (proceeding in this way, through the system (A9.18, A9.19) avoids making Lipschitz conditions on f).

From (A9.12) in Sub-lemma A9.1 with $\underline{b} = b$, we find:

$$\begin{aligned} & \int_{\alpha(b;\eta_k)}^{w} F\left(v\right)^n \frac{d}{dv}\rho\left(v, \varphi\left(v\right)\right) \\ \leq & \rho\left(w, \varphi\left(w\right)\right) - \beta\left(w\right) \\ \leq & 3\varepsilon, \end{aligned}$$

for all $k \geq 1$ such that $\eta^* - \delta < \eta_k$.

From $\alpha(b;\eta_k) < v' < \alpha^*(\underline{b}(\eta^*)) < w$, we then find:

$$\int_{v'}^{\alpha^*(\underline{b}(\eta^*))} F(v)^n \frac{d}{dv}\rho(v,\varphi(v)) \leq 3\varepsilon,$$

and, from Lemma A9.1 and the inequality $\frac{d}{dv}\rho(v,\varphi(v)) \geq \frac{\partial}{\partial v}\rho(v,\varphi(v))$, we then obtain:

$$(\alpha^*(\underline{b}(\eta^*)) - v')\, L \leq 3\varepsilon,$$

where L is a strictly positive lower bound of $\frac{\partial}{\partial v}\rho(v,w)$ over $[v',d]^2$. This inequality must hold for all $\varepsilon > 0$, which is cl;early impossible since the LHS is a strictly positive constant. We have proved

$$\lim_{\eta \to_< \eta^*} \alpha(b;\eta) = \underline{b}(\eta^*),$$

for all b in $(c, \underline{b}(\eta^*))$. From the inequalities (A9.15), we then find $\lim_{\eta\to_<\eta^*} \rho(\alpha(b;\eta), \gamma(b;\eta)) = \underline{b}(\eta^*)$ and, consequently (because ρ is continuous and strictly increasing), $\lim_{\eta\to_<\eta^*} \gamma(b;\eta) = \underline{b}(\eta^*)$. ||

Lemma A9.10:

$$\underline{b}(\eta^*) = c$$

and the solution of (A9.2-A9.4) for the value η^* of the parameter satisfies the boundary conditions $\beta(c) = \delta(c) = c, \beta(d) = \delta(d)$.

Proof: Suppose that, as in Lemma A9.9, $\underline{b}(\eta^*) > c$. For $\eta \leq \eta^*$, consider the function below:

$$\ln(\underline{b}(\eta^*) - b) + n \ln F(\alpha(b;\eta)), \text{ (A9.20)}$$

which is defined for b in $(c, \underline{b}(\eta^*)) \subseteq (\underline{b}(\eta), \underline{b}(\eta^*))$. Let b' be in this interval $(c, \underline{b}(\eta^*))$. Let ε be a (small) strictly positive number. Since, from Lemma A9.9, we have $\lim_{\eta\to_<\eta^*} \rho(\alpha(b;\eta), \gamma(b;\eta)) = \lim_{\eta\to_<\eta^*} \alpha(b;\eta) = \underline{b}(\eta^*)$, for all b in $(c, \underline{b}(\eta^*))$, there exists $\delta > 0$ such that

$$\begin{aligned} |F(\alpha(\underline{b}(\eta^*) - \varepsilon; \eta)) - F(\underline{b}(\eta^*))| &< \varepsilon, \\ \rho(\alpha(b'; \eta), \gamma(b'; \eta)) &> \underline{b}(\eta^*) - \varepsilon/2, \text{ (A9.21)} \end{aligned}$$

for all η such that $\eta^* - \delta < \eta < \eta^*$.

From (A9.3), the derivative, with respect to b, $\frac{-1}{\underline{b}(\eta^*)-b} + n\frac{d}{db}\ln F(\alpha(b;\eta))$ of the function (A9.20) is equal to:

$$\frac{-1}{\underline{b}(\eta^*) - b} + \frac{1}{\rho(\alpha(b;\eta), \gamma(b;\eta)) - b},$$

and consequently, we have:

$$\begin{aligned} & \ln(\underline{b}(\eta^*) - b') + n \ln F(\alpha(b';\eta)) \\ \leq\ & \ln \varepsilon + n \ln\{F(\underline{b}(\eta^*)) + \varepsilon\} \\ & - \int_{b'}^{\underline{b}(\eta^*)-\varepsilon} \left(\frac{1}{\rho(\alpha(b;\eta), \gamma(b;\eta)) - b} - \frac{1}{\underline{b}(\eta^*) - b} \right) db, \text{(A9.22)} \end{aligned}$$

for all η such that $\eta^* - \delta < \eta < \eta^*$.

Since $\underline{b}(\eta^*) - b \geq \varepsilon$ and, from (A9.21), $\rho(\alpha(b;\eta), \gamma(b;\eta)) - b \geq \rho(\alpha(b';\eta), \gamma(b';\eta)) - (\underline{b}(\eta^*) - \varepsilon) \geq \varepsilon/2$ over the integration interval in (A9.22), we have the following bound over this interval:

$$\left| \frac{1}{\rho(\alpha(b;\eta), \gamma(b;\eta)) - b} - \frac{1}{\underline{b}(\eta^*) - b} \right| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon}.$$

We may thus apply Lebesgue convergence theorem, for example, and we find $\lim_{\eta \to_< \eta^*} \int_{b'}^{\underline{b}(\eta^*)-\varepsilon} \left(\frac{1}{\rho(\alpha(b;\eta),\gamma(b;\eta))-b} - \frac{1}{\underline{b}(\eta^*)-b} \right) db = 0$ and, consequently,:

$$\begin{aligned} & \ln(\underline{b}(\eta^*) - b') + n \ln F(\underline{b}(\eta^*)) \\ =\ & \lim_{\eta \to_< \eta^*} \{\ln(\underline{b}(\eta^*) - b') + n \ln F(\alpha(b';\eta))\} \\ \leq\ & \ln \varepsilon + n \ln\{F(\underline{b}(\eta^*)) + \varepsilon\}. \end{aligned}$$

Since this inequality holds for all $\varepsilon > 0$, we obtain $\ln\left(\underline{b}(\eta^*) - b'\right) + n \ln F\left(\underline{b}(\eta^*)\right) = -\infty$ or, equivalently, $\left(\underline{b}(\eta^*) - b'\right) F\left(\underline{b}(\eta^*)\right) = 0$, which is impossible, since $\underline{b}(\eta^*) > 0$ and $b' < \underline{b}(\eta^*)$. We have proved that $\underline{b}(\eta^*) > c$ is impossible, that is, we have proved the equality $\underline{b}(\eta^*) = c$. ||

Lemma A9.11: *There cannot exist two different values of the parameter η such that the corresponding solutions to (A9.2-A9.4) are defined over $(c, \eta]$ and such that $\alpha(c) = \gamma(c) = c$.*

Proof: Suppose there exists two such values η' and $\widetilde{\eta}$, with $\eta' < \widetilde{\eta} < d$. Let α', δ' and $\widetilde{\alpha}, \widetilde{\delta}$ the corresponding solutions to (A9.2-A9.4). Then, $\beta' = \alpha'^{-1}, \varphi' = \delta'\alpha'$, and $\widetilde{\beta} = \widetilde{\alpha}^{-1}, \widetilde{\varphi} = \widetilde{\delta}\widetilde{\alpha}$ are solutions to (A9.16, A9.17) with initial condition $\varphi(d) = d, \beta(d) = \eta$.

From (A9.16), we have $\frac{d}{d\ln F}\varphi'(d) = \frac{d}{d\ln F}\widetilde{\varphi}(d) = \frac{1}{h(d)}$. Moreover, from (A9.16, A9.17) and the differentiability of $\rho(v, \varphi(v))$ and $\rho(v, \widetilde{\varphi}(v))$; $\frac{d}{d\ln F}\varphi'$, $\frac{d}{d\ln F}\widetilde{\varphi}$, $\frac{d}{d\ln F}\beta$, $\frac{d}{d\ln F}\widetilde{\beta}$ are differentiable and, by differentiating (A9.16), we find:

$$
\begin{aligned}
& \frac{d}{dv}\left(\frac{d}{dnF}\varphi'(v)\right)_{v=d} - \frac{d}{dv}\left(\frac{d}{d\ln F}\widetilde{\varphi}(v)\right)_{v=d} \\
= \; & \frac{1 - \frac{d}{dv}\rho(v, \widetilde{\varphi}(v))_{v=d}}{d - \widetilde{\eta}} - \frac{1 - \frac{d}{dv}\rho(v, \varphi'(v))_{v=d}}{d - \eta'} \\
= \; & \left(1 - \frac{1 + (f(d)/h(d))}{2}\right)\left(\frac{1}{d - \widetilde{\eta}} - \frac{1}{d - \eta'}\right) \\
> \; & 0,
\end{aligned}
$$

since, from our assumption (iv), $\frac{f(d)}{h(d)} < 1$. Consequently, there exists $\varepsilon > 0$, such that $\varphi'(v) > \widetilde{\varphi}(v)$ and, from the initial condition $\beta(d) = \eta$, $\beta'(v) < \widetilde{\beta}(v)$, for all v in $(d - \varepsilon, d)$.

Let $\underline{v}$ be defined as follows:

$$
\underline{v} = \inf\left\{v \in [c, d] \,|\varphi'(w) > \widetilde{\varphi}(w) \text{ and } \beta'(w) < \widetilde{\beta}(w), \text{ for all } w \text{ in } (v, d)\right\}.
$$

From the previous paragraph, $\underline{v} \leq d - \varepsilon$. Suppose $\underline{v} > c$. Since φ',β' and $\widetilde{\varphi}, \widetilde{\beta}$ are distinct solutions of the same differential system, the equalities $\varphi'(\underline{v}) = \widetilde{\varphi}(\underline{v})$ and $\beta'(\underline{v}) = \widetilde{\beta}(\underline{v})$ cannot both hold. Assume first $\varphi'(\underline{v}) > \widetilde{\varphi}(\underline{v})$ and $\beta'(\underline{v}) = \widetilde{\beta}(\underline{v})$. From (A9.17), $\frac{d}{dv}\beta(\underline{v}) > \frac{d}{dv}\widetilde{\beta}(\underline{v})$, which is impossible since $\beta'(w) < \widetilde{\beta}(w)$ holds true over $(\underline{v}, d)$. Assume next $\varphi'(\underline{v}) = \widetilde{\varphi}(\underline{v})$ and $\beta'(\underline{v}) < \widetilde{\beta}(\underline{v})$. The factor between braces in (A9.16) can be rewritten as:

$$n - 1 - (n-2)\left[1 - \frac{\rho(v, \varphi(v)) - v}{\rho(v, \varphi(v)) - \beta(v)} \frac{H(\rho(v, \varphi(v)))}{H(\varphi(v))}\right],$$

and hence is increasing in $\beta(v)$. Consequently, $\frac{d}{dv}\varphi'(\underline{v}) < \frac{d}{dv}\widetilde{\varphi}(\underline{v})$, which is impossible since $\varphi'(w) > \widetilde{\varphi}(w)$ holds true over $(\underline{v}, d)$.

We have proved $\underline{v} = c$, which implies $\varphi'(w) > \widetilde{\varphi}(w)$, for all w in (c, d). From (A9.13) in Sublemma 1, we then have:

$$\begin{aligned} & \beta'(d) \\ = & \int_c^d \rho(v, \varphi'(v)) \frac{d}{dv} F(v)^n \\ > & \int_c^d \rho(v, \widetilde{\varphi}(v)) \frac{d}{dv} F(v)^n \\ = & \widetilde{\beta}(d) \end{aligned}$$

and $\beta'(d) > \widetilde{\beta}(d)$. However, this is impossible since, from the initial condition at d, $\beta'(d) = \eta' < \widetilde{\eta} = \widetilde{\beta}(d)$. ||

Proof of Theorem 4

(i) follows from Lemmas A9.10 and A9.11; (ii) from Lemma A9.5; and (iii) from Appendix 8.

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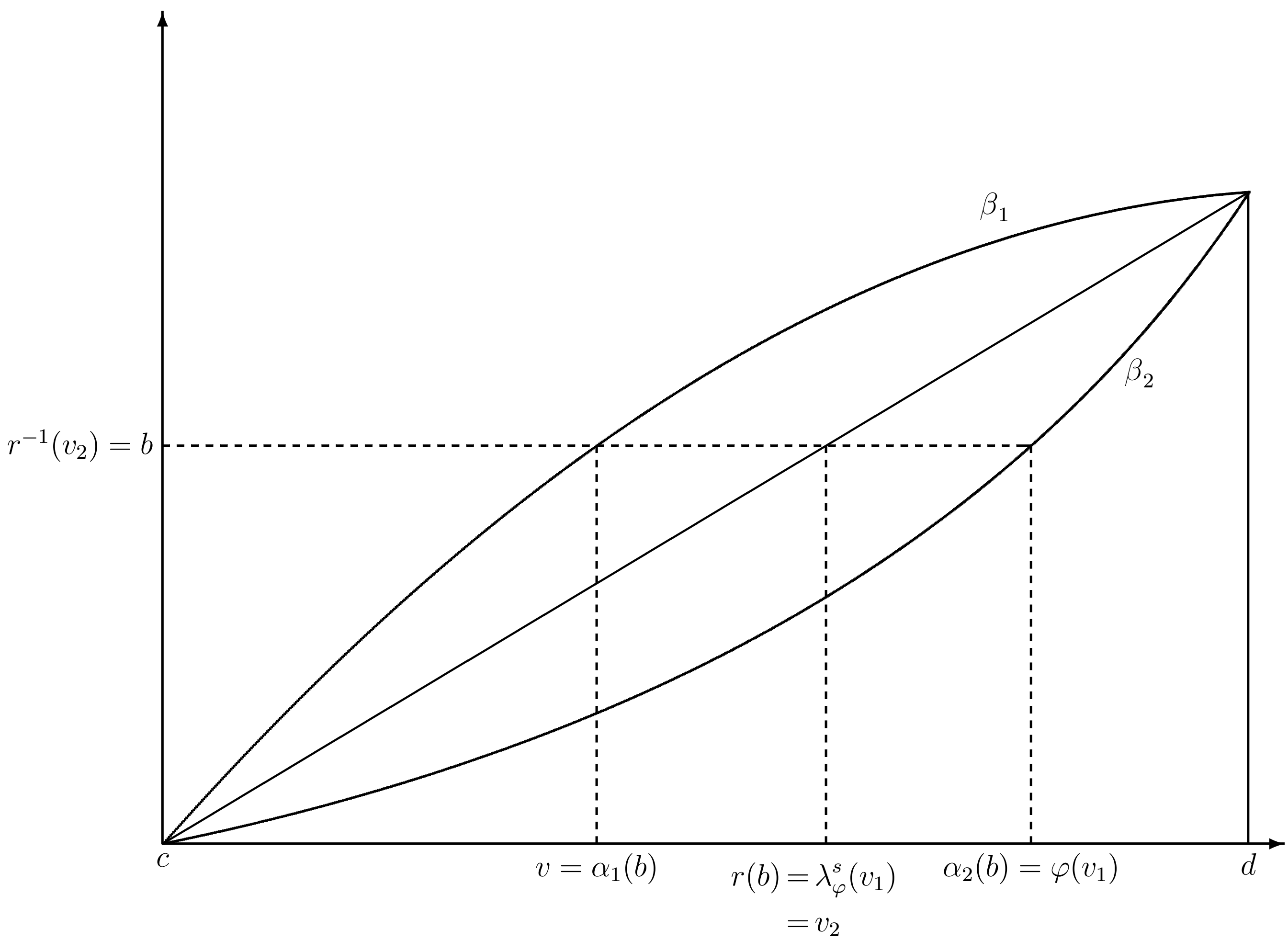

β_1
β_2
$r^{-1}(v_2) = b$
c
$v = \alpha_1(b)$
$r(b) = \lambda_\varphi^s(v_1)$
$= v_2$
$\alpha_2(b) = \varphi(v_1)$
d

Figure 1

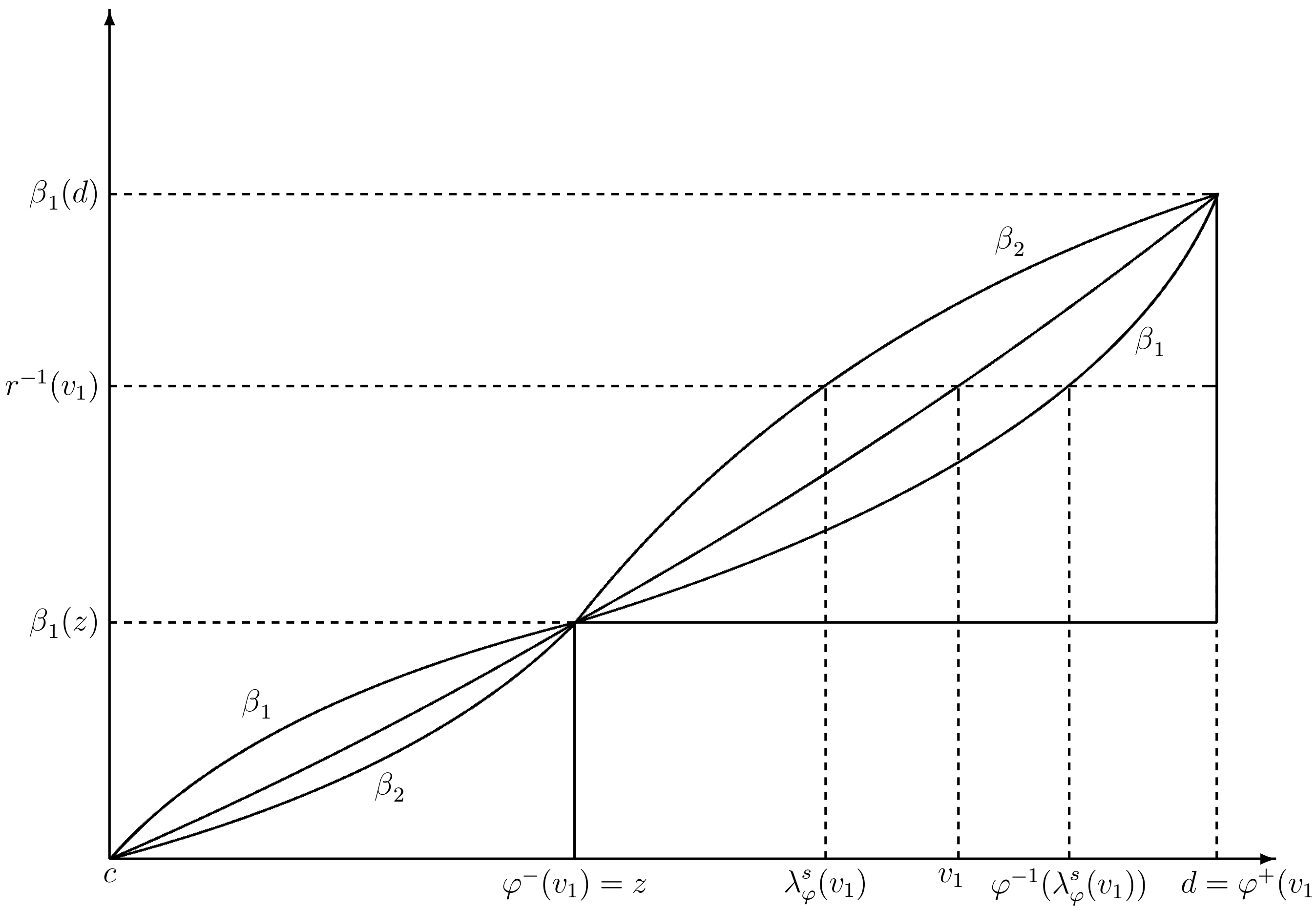

$\beta_1(d)$
$r^{-1}(v_1)$
$\beta_1(z)$
β_2
β_1
β_1
β_2
c
$\varphi^-(v_1) = z$
$\lambda^s_\varphi(v_1)$
v_1
$\varphi^{-1}(\lambda^s_\varphi(v_1))$
$d = \varphi^+(v_1)$

Figure 2

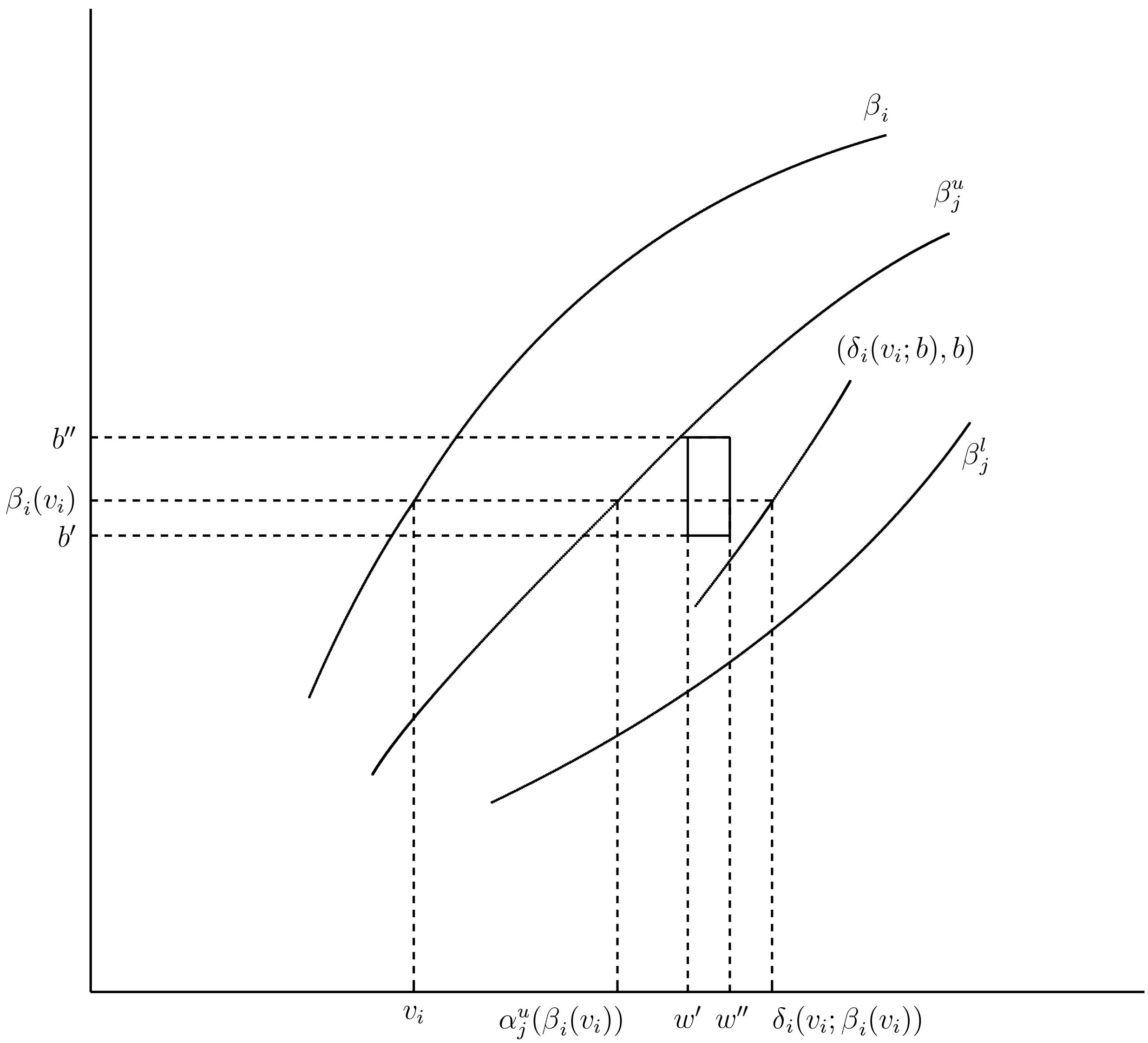
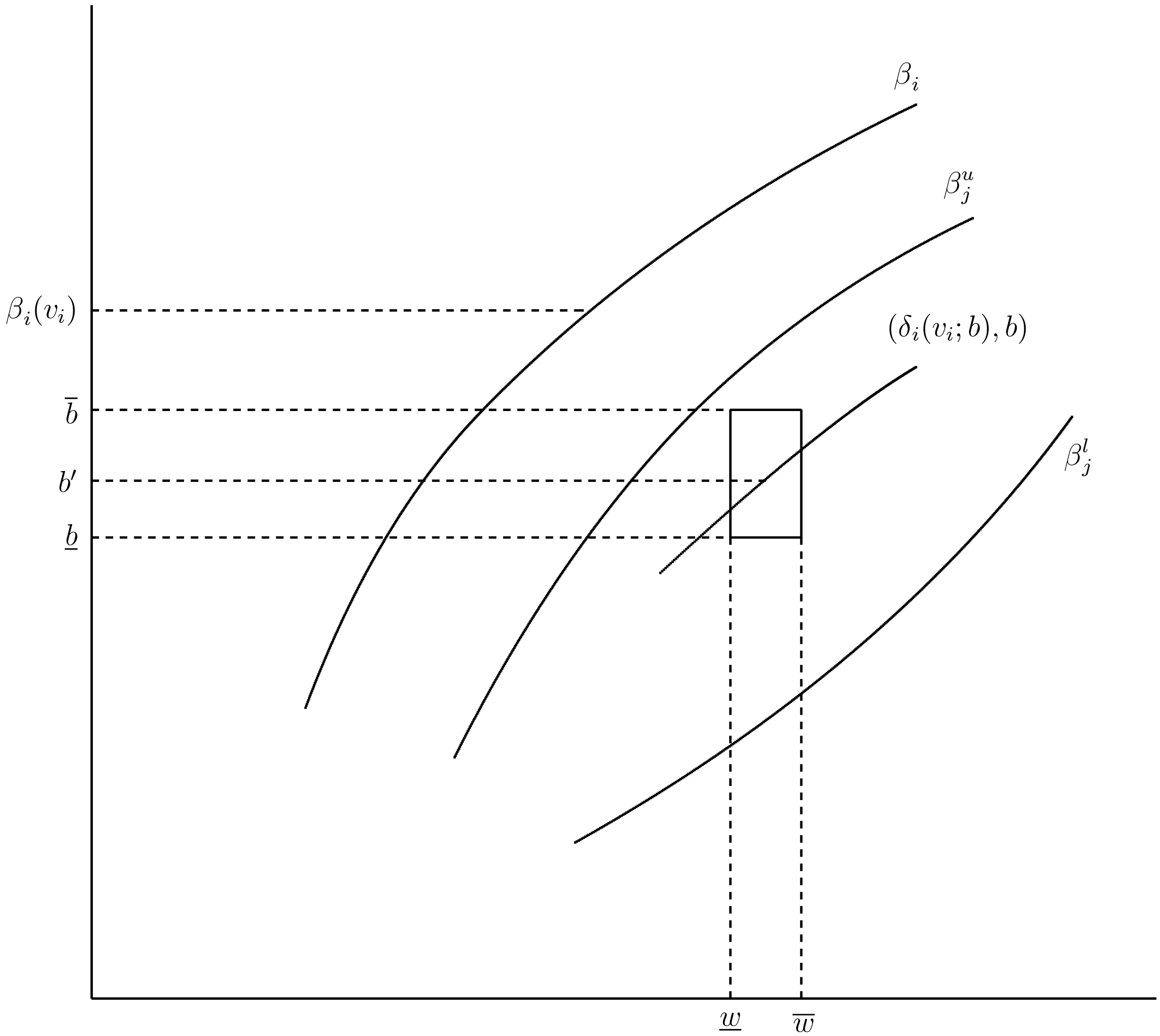

β_i
β_j^u
$(\delta_i(v_i;b),b)$
β_j^l
b''
$\beta_i(v_i)$
b'
v_i
$\alpha_j^u(\beta_i(v_i))$
w'
w''
$\delta_i(v_i;\beta_i(v_i))$

Figure A1



β_i
β_j^u
$(\delta_i(v_i;b),b)$
β_j^l
$\beta_i(v_i)$
$\overline{b}$
b'
$\underline{b}$
$\underline{w}$
$\overline{w}$

Figure A2

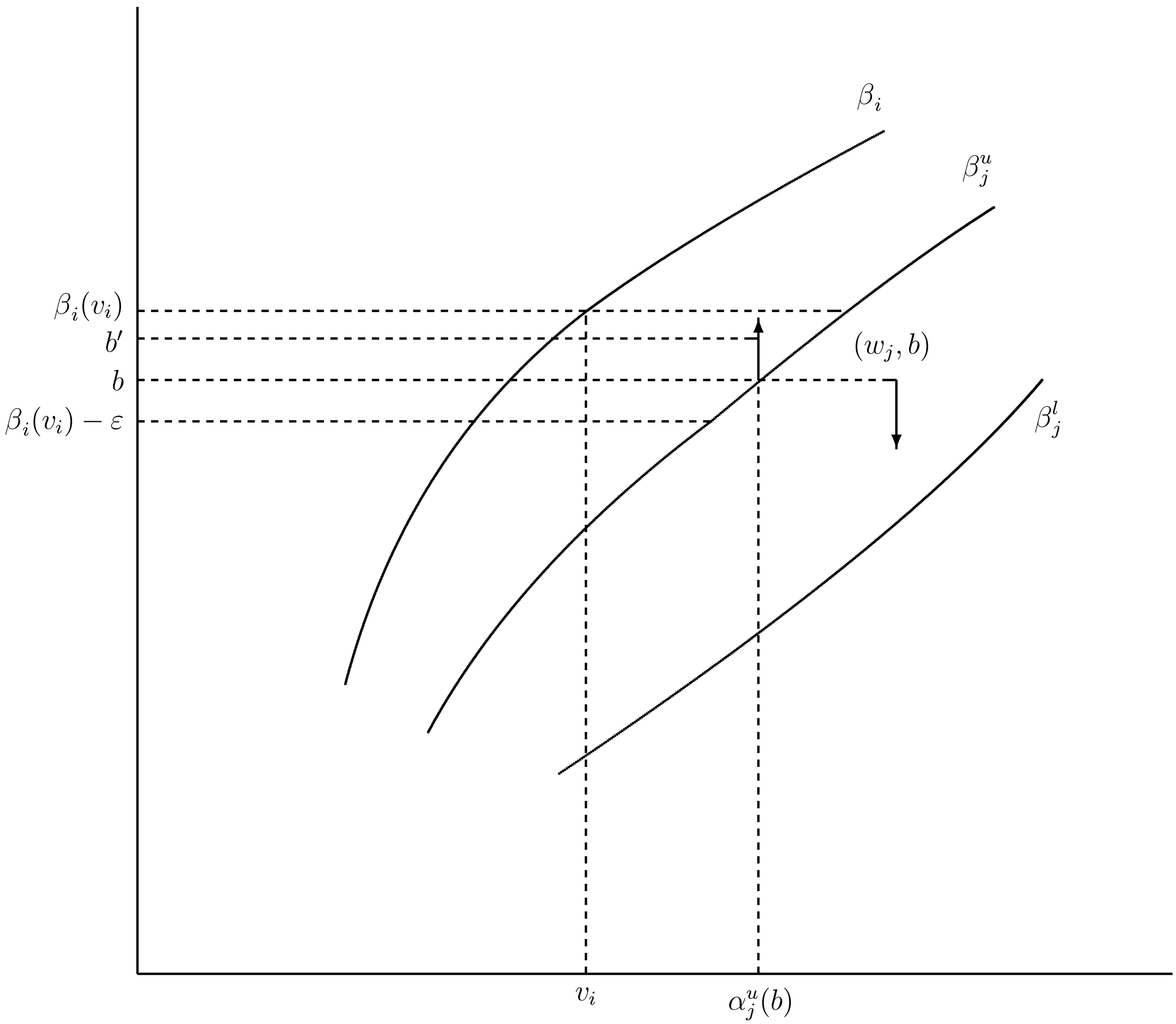

β_i
β_j^u
$\beta_i(v_i)$
b'
(w_j, b)
b
β_j^l
$\beta_i(v_i) - \varepsilon$
v_i
$\alpha_j^u(b)$

Figure A3

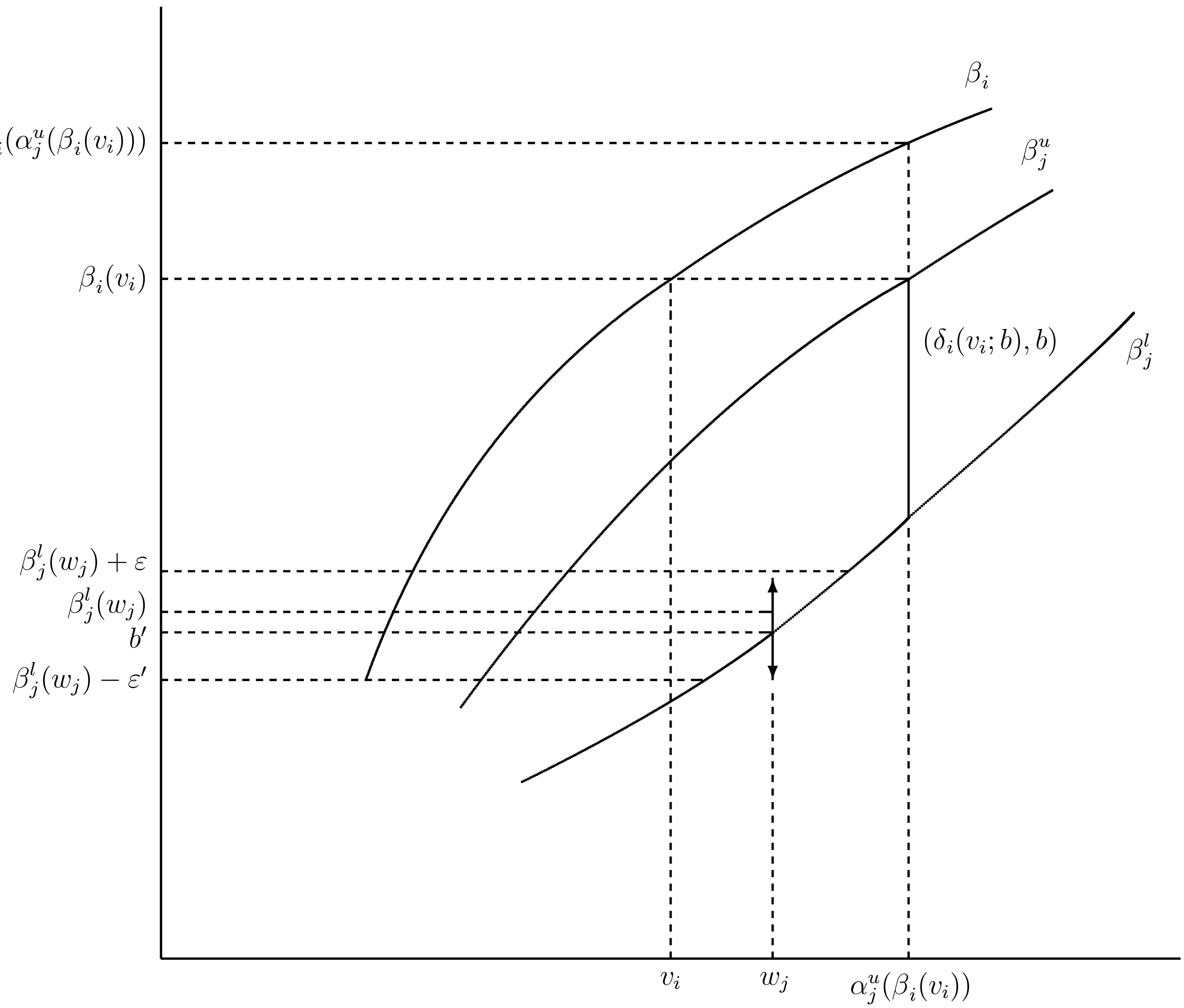

β_i
β_j^u
β_j^l
$(\alpha_j^u(\beta_i(v_i)))$
$\beta_i(v_i)$
$(\delta_i(v_i; b), b)$
$\beta_j^l(w_j) + \varepsilon$
$\beta_j^l(w_j)$
b'
$\beta_j^l(w_j) - \varepsilon'$
v_i
w_j
$\alpha_j^u(\beta_i(v_i))$

Figure A4