# The Ordered Qualitative Model For Credit Rating Transitions

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**Abstract**

Information on the expected changes in credit quality of obligors is contained in credit migration matrices which trace out the movements of firms across ratings categories in a given period of time and in a given group of bond issuers. The rating matrices provided by Moody's, Standard &Poor's and Fitch became crucial inputs to many applications, including the assessment of risk on corporate credit portfolios (CreditVar) and credit derivatives pricing. We propose a factor probit model for modeling and prediction of credit rating matrices that are assumed to be stochastic and driven by a latent factor. The filtered latent factor path reveals the effect of the economic cycle on corporate credit ratings, and provides evidence in support of the PIT (point-in-time) rating philosophy. The factor probit model also yields the estimates of cross-sectional correlations in rating transitions that are documented empirically but not fully accounted for in the literature and in the regulatory rules established by the Basle Committee.

**Keywords:** Credit Rating, Migration, Migration Correlation, Credit Risk, Probit Model, Latent Factor, Business Cycle.

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# 1 Introduction

Agency rating of credit quality provides an important measure of credit risk. The corporate ratings are routinely performed and updated by agencies such as the Moody's, Standard &Poor's and Fitch. The records of past and present ratings document the movements of a firm from one rating category to another, and arise as direct indicators of risk dynamics. Information on the future expected changes in credit quality is contained in the credit migration matrices which trace out the movements of firms in a given group of bond issuers across ratings categories in a given period of time. This information is used for computing the risk on portfolios of corporate credits, including the CreditVar, and for credit derivatives pricing. Therefore, the methods used in the banking sector for modeling and prediction of risk migration matrices have an important effect on the performance of internal credit pricing models (CPM) and the functioning of risk monitoring systems.

Recent literature has explored the variation of credit rating matrices in time [see e.g. Altman, Kao (1991)] and investigated the links between the rating dynamics and the underlying macroeconomic conditions [see Nickell et al. (2000), Bangia et al. (2002) and Rosch (2005)]. A common feature of recent contributions to the literature is the dependence of credit migration dynamics on observable macroeconomic variables such as, for example, a proxy of the stage of the business cycle [see Nickell et al. (2000), Bangia et al. (2002)], a riskfree interest rate or the unemployment rate [Rosch (2005)]. Our study explores the dynamics of credit rating matrices from a different perspective. We do not assume a priori that the rating matrices are driven by a particular, observed variable. Instead, we introduce an unobservable (latent) factor that allows the data to reveal their own intrinsic driving process. The estimated path of the factor can then be compared to relevant macroeconomic variables and interpreted. Our result confirms the link between credit quality changes and the underlying state of the economy, but points to the continuously valued GDP growth rate as the leading process rather than to the business cycle indicator. In the context of the on-going debate on the PIT (point-in-time) versus TTC (through-the-cycle) rating philosophy our findings clearly support the PIT approach.

The analysis is carried out in the framework of an ordered qualitative variable model [see the Basle Committee documents, Cheung (1996), Nickell et al (2000)]. This approach explicitly allows for discreteness of ratings and accommodates naturally the ordering from low to high credit quality.

The paper is organized as follows. Section 2 introduces the notation and basic assumptions. Section 3 presents the time-varying parameter probit and stochastic factor probit models for rating transition probabilities. In Section 4, we compare both models and emphasize the important features of the stochastic factor probit. One of them is cross-sectional correlation in credit quality migration among obligors. The correlation arises from joint migrations when several firms are jointly down- or up-graded. Statistical inference is discussed in Section 5 for both specifications. In particular, we propose the simulated maximum likelihood as a consistent and efficient estimation method for the stochastic factor model. Practitioners may however find this method complicated and possibly time consuming. Therefore, as an alternative approach, we develop a simplified estimation method based on a linear approximation of the nonlinear factor model. It is not only easier to implement, but also preserves the consistency and asymptotic efficiency properties of the estimator due to the large cross-sectional dimension of the dataset. In Section 6, the methodology is applied to data on aggregate transition frequencies reported by the Standard & Poor's. Estimation of both types of probit models, and various summary statistics produced by the models, such as " average risk" and "risk volatility" per rating category, are discussed in the sequel. A one factor probit model is used in Section 7 to predict future ratings and correlations in rating transitions at different horizons. We observe that migration correlations depend on the time it takes to migrate, on the current economic environment and on the ratings in which the migration starts and in which it ends. These dependencies are not taken into account in the most recent document issued by the Banking Supervisory Committee in Basle. Also, we find that the estimated correlations are different, and in most cases much lower than the fixed input values provided by the Basle Committee in 2003 to banks for computation of the minimum capital reserve. This suggest that the current capital reserves are likely overestimated, and the cost of corporate credit is artificially increased. Section 8 concludes.

# 2 Specification of Credit Rating Dynamics

## 2.1 Observed transition matrices

The object of our analysis is a sequence of square matrices with positive elements formed by the frequencies of transitions between states. The dimension of each matrix can be of the order of 8 by 8 up to 10 by 10, depending on the number of states considered in the analysis [1]. The states, denoted $k,\ k = 1, \cdots, K$ represent credit ratings, assigned by a credit rating agency, such as the Standard & Poor's, for example, for a fixed period of time. Accordingly, $K$ is equal to the number of credit categories including default [2]. In our study $k = 1$ indicates the highest rating while $k = K$ indicates default. The rating matrices are updated and reported, at a fixed frequency of one year. Therefore a typical sequence in an empirical study comprises 10-20 matrices. The sequence of credit rating matrices is obtained by aggregating the individual firm credit rating histories.

The entries in each credit migration matrix are sample frequencies of transitions from one category to another which occurred in a given year. They are positive and sum up to one in each row (or column, depending on the convention). Since sample transition frequencies approximate the transition probabilities from lower to higher credit ratings and vice-versa, modelling and prediction of credit migration matrices is a crucial element in risk management.

## 2.2 Individual credit rating histories

Data on individual rating histories are not always available. However, prediction of future ratings for a given set of companies is the ultimate goal of any credit risk analysis. Formally, an individual rating history of firm $i$ denoted $(Y_{i,t},\ t = 1, \cdots, T),\ i = 1, \cdots, n$ corresponds to a qualitative process with state space $\{1, \cdots, K\}$ observed at times $t = 1, .., T$.

A basic approach which underlies the practice of rating agencies, when they aggregate individual rating histories into transition frequencies relies on the following assumption:

[1] For credit ratings, the regulators require the number of states to lie between 8 and 10 (including default).

[2] Standard & Poor's use the following ratings: AAA, AA, A, BBB, BB, B, CCC, CC, C, D. The three C-type of categories are often aggregated into a single category CCC. AAA is the best credit quality rating, C is the worst one, and D stands for default.

**Assumption A1**: The individual credit rating histories $(Y_{i,t},\ t = 1, \cdots, T),\ i = 1, \cdots, n$ are independent and identically distributed across the firms; Moreover, each individual credit rating history follows a heterogeneous Markov chain.

This means that the most recent individual rating contains all information on the rating history of a given firm, and that the transition probabilities can be time varying.

Under Assumption A1, the joint dynamics of rating histories of all firms is characterized by the sequence of transition matrices $P_t,\ t = 1, \cdots, T$. The matrix $P_t$ is a $(K, K)$ matrix. Its elements provide the transition probabilities from rating $l$ to another rating $k$ between dates $t$ and $t+1$,

$$p_{k,l,t} = P[Y_{i,t+1} = k | Y_{i,t} = l], \quad \forall\, k, l, t. \tag{1}$$

The transition probabilities are the same for different individuals, which follows from the cross-sectional homogeneity assumption on the population of firms. They are non-negative and sum up to one by columns. Since the state $K$ is equivalent to default, which is an absorbing state, the number of probabilities to be estimated and predicted is $(K-1)^2$.

The homogeneity of firms in the population, or formally the identical distribution of individual rating histories, imposed in A1 is often questioned in the literature [3] even when the ratings concern firms that belong to the same industrial sector and country of origin [4]. However, the independence and identity of distributions in Assumption A1 is conditional on the sequence of transition matrices $(P_t)$, and on the factors influencing the transition probabilities. These factors can be a) observable, as in the McKinsey methodology [see Wilson (1997)] and CreditRisk+ [see Credit Suisse Financial Products (1997)], or b) unobservable variables whose presence creates cross-sectional correlation between risk, and leads to serial dependence in risk, when these factors are integrated out.

The discreteness of ratings and the fact that the ratings possess a natural ordering from low to high credit quality suggest the use of an ordered qualitative dependent variable model. Two types of dynamic ordered qualitative dependent variable models are proposed in the next section. In the first model, the credit ratings of each company are determined

[3]see Gordy (2003), Gagliardini, and Gouriéroux(2005a) for a discussion of the population homogeneity assumption.

[4]The Basle Committee suggests the use of three segmentation criteria that are the industrial sector, country of origin and credit quality rating for determining homogeneous subpopulations.

by a latent score function with non-stochastic time varying parameters. In the second approach, the parameters of the latent score are stochastic and driven by another latent variable, called the factor.

# 3 The Ordered Qualitative Variable Model

The ordered qualitative variable model is based on the assumption that individual qualitative ratings are determined from an unobserved continuously valued score. Usually, the latent score is an increasing function of estimated default probability, for each firm at every date [5]. Let us denote $S_{i,t}$ the value of the score for firm $i$ at time $t$. The qualitative rating is obtained by discretizing the score values. More precisely, let us introduce a partition $a_{1,t} < \cdots < a_{K-1,t}$ of admissible values of the score. The observed rating is defined by:

$$Y_{i,t} = k, \quad \text{if and only if } \; a_{k-1,t} < S_{i,t} < a_{k,t}, \qquad (2)$$

where by convention, $a_{0,t} = -\infty$ and $a_{K,t} = \infty$. Relation (2) defines the link between the observable endogenous credit rating $Y_{i,t}$ of firm $i$ and the latent score $S_{i,t}$ at time $t$.

## 3.1 Time-varying parameter probit model

The model is obtained by specifying the (conditional) distribution of the quantitative score:

**Assumption A2** : The individual score processes are independent, identically distributed across the population of firms. The conditional distribution of the score of a firm $S_{i,t+1}$, given the lagged score values depends on the past through the most recent qualitative rating only.

The dynamics of the individual score is:

$$S_{i,t+1} = m_{l,t} + \sigma_{l,t} u_{i,t+1}, \text{ if } \; Y_{i,t} = l, \; l = 1, \cdots, L, \qquad (3)$$

where $u_{i,t}$ are i.i.d. variables with the same cumulative density function (cdf) $F$. The subscripts indicate that the individual conditional drift and volatility of the score depend

[5] See, Gourieroux, Jasiak (2006) on credit scoring.

on the current rating, as well as on time [6].

Assumption A2 implies Assumption A1. Under Assumption A2 the transition probabilities are:

$$\begin{aligned} p_{k,l,t} &= P[Y_{i,t+1} = k | Y_{i,t} = l] \\ &= P[a_{k-1,t} < S_{i,t+1} < a_{k,t} | Y_{i,t} = l] \end{aligned}$$

or:

$$p_{k,l,t} = F(\frac{a_{k,t} - m_{l,t}}{\sigma_{l,t}}) - F(\frac{a_{k-1,t} - m_{l,t}}{\sigma_{l,t}}). \quad (4)$$

For example, $p_{K,l,t} = 1 - F(\frac{a_{K-1,t} - m_{l,t}}{\sigma_{l,t}})$ is the probability that a firm rated "$l$" makes a transition to "K" that is to default. Similarly, $p_{1,l,t} = F(\frac{a_{1,t} - m_{l,t}}{\sigma_{l,t}})$ is the probability of migration to the best credit quality category "1" (AAA in the S&P's).

The parameters of the model are $a_{k,t}, m_{l,t}, \sigma_{l,t}, k, l = 1, \cdots, K, t = 1, \cdots, T$. The parameters are identifiable up to some time dependent linear affine transformation on $a, m$ and $\sigma$. Thus we cannot distinguish $a_{k,t}, m_{l,t}, \sigma_{l,t}$ from $a^*_{k,t} = \alpha_t a_{k,t} + \beta_t$, $m^*_{l,t} = \alpha_t m_{l,t} + \beta_t$ and $\sigma^*_{l,t} = \alpha_t \sigma_{l,t}$, for any $\beta_t$ and $\alpha_t$, $\alpha_t > 0$. In the sequel we use the following identifying restrictions [7]: $a^*_{1,t} = 0, \sigma^*_{1,t} = 1, \ \forall t$. Under the identifying restrictions, the new parameters are related to the initial ones as follows:

$$a^*_{k,t} = \frac{a_{k,t} - a_{1,t}}{\sigma_{1,t}}, \ m_{l,t}* = \frac{m_{l,t} - a_{1,t}}{\sigma_{1,t}}, \ \sigma^*_{l,t} = \frac{\sigma_{l,t}}{\sigma_{1,t}}, \ \forall t.$$

In particular, the hypotheses such as "thresholds constant in time", "variance constant in time", or hypotheses concerning the distribution of the score are not identifiable. Under the ordered qualitative model, the $(K-1)^2$ different transition probabilities depend on a number of parameters equal to $3(K-1)$.

The ordered probit model is obtained when the error variable $u_{i,t}$ is assumed standard normal, and the cdf $F$ is replaced by $\Phi$, the cdf of the standard normal. The normality assumption is frequently encountered in the applied and theoretical literature [see e.g. Merton(1974), the documentation of the Basle Committee, Gupton, Finger, Bhatia (1997), Nickell et al. (2000), Bangia et al. (2002), Albanese, Chen (2003), Albanese et al. (2003), Rosch (2005)].

[6] As in Assumption A1, the independence and Markov of order 1 property have to be interpreted as conditional on the drift and volatility values $m_{l,t}, \sigma_{l,t}, t = 1, ..., T$.

[7] These identifying restrictions are quite specific. Indeed, we have as many ordered qualitative models subject to cross-parameter restrictions as the number of initial states.

### 3.2 Stochastic parameter (factor) probit model

In the previous model, the transition matrices are pre-specified functions of time varying parameters $p_{klt} = p_{kl}(\theta_t)$, where $\theta_t$ comprises all parameters of the model at date $t$. Let us now extend the model to allow for stochastic transition matrices [8]. This is accomplished by writing the time dependent parameters $a_{k,t}, m_{l,t}, \sigma_{l,t}$ of the score function as time independent functions of (unobservable) stochastic dynamic factors ($F_t$). In this setup, the transition matrices depend on fixed parameters $\tilde{\theta}$ and a time varying, stochastic component $F_t$: $p_{klt} = p_{kl}(F_t; \tilde{\theta})$.

Not all parameters of the score need to be stochastic. For example, we can assume that $\sigma_{l,t} = \sigma_l$, independent of $t$, $a_{k,t} = a_k$, independent of $t$, and consider a linear factor representation for the latent drifts:

$$m_{l,t} = \sum_{i=1}^{M} \gamma_{l,i} F_{i,t} + \delta_l, \tag{5}$$

where the $M$ factors satisfy a Gaussian Vector Autoregressive (VAR) model:

$$F_{t+1} = AF_t + \eta_t, \tag{6}$$

and the error terms $(\eta_t)$ are i.i.d. standard normal vectors. In this case, $\tilde{\theta} = \{\sigma_k, a_k, \gamma_{k,i}, \delta_{k,i}\ k = 1, \cdots, K,\ i = 1, \cdots, M\}$. Under the normality assumption on the error terms, the model becomes a serially correlated factor probit.

The parameters of the factor probit to be estimated are (i) the parameters $\tilde{\theta}$ defining the probabilities, (ii) the parameter (matrix of parameters) $A$, which characterizes the factor dynamics. By introducing the factor representation (4)-(5)-(6), we replace the $(K-1)T$ parameters of time dependent latent means by $(K-1)(M+1) + M^2$ parameters. Some sensitivity coefficients $\gamma$ can be close to zero for some particular rating categories, such as AAA, and significantly different from zero for a bad credit quality category, for example. In such a case the associated factor, $F_{i,t}$, captures the dynamics of bad credit quality ratings only.

In the factor ordered qualitative model, the current rating depends on the last observed rating and the last factor value when the factor values are integrated out. The rating

[8] In this respect, we extend to a multivariate setup the stochastic intensity model, (also called Cox' model) widely used for default modeling [see e.g. Lando (1998), Duffie and Singleton (1999), Yu (2002)].

histories are no longer Markov process of order 1 and become cross-sectionally dependent [see e.g. Frey, McNeil (2001)]. Each current rating is influenced by all past ratings, including the ratings of other firms which provide information on the (unobservable) past factor values. Note that individual rating histories are stationary whenever the factor process is stationary. The stationarity condition requires that the eigenvalues of matrix $A$ have modulus less than one, for a multivariate factor process.

The rationale for introducing the serially correlated random factor is inspired by findings in recent literature [9]. Bangia et al (2002) argue that credit migration matrices depend on a macroeconomic variable. In their approach, the factor is observable and is approximated by a sequence of state of the economy indicators (GDP recessions and expansions). A similar approach has also been implemented by Nickell et al. (2000), who proposed a probit model with observed explanatory variables, and a time dependent qualitative variable related to business cycle and based on the GDP growth rate by country. The last one distinguishes between high, low and middle range growth rate recorded in the sampling period. The model with observable factors is simple to implement from the statistical point of view. However, it can lead to misspecification if the factor variable is not well selected, or is measured with error. For instance, the importance of the U.S. business cycle can be questioned in a set of firms, which include almost 30% of foreign firms, and can comprise obligors from industrial sectors with different cycles. Also, one may argue that other observables, such as the duration of the cycle may be better suited. By introducing unobservable factors we avoid possible misspecification and additional noise, and allow the data to reveal its own intrinsic driving process. In our approach we don't impose any prior economic interpretation of the factors, but instead we are able to provide it once the latent factor process is estimated.

Moreover, the transition models are used for forecasting of future ratings and, in particular, for computing the minimum capital reserve required to hedge a given corporate credit portfolio (the so-called CreditVaR). For this purpose, it is necessary to predict fu-

[9] In recent literature, there exist very few rating models with an unobservable factor [see e.g. Gordy(2003), Rosch (2005)] although such an approach was suggested by the Basle Committee (2003). However, the Basle Committee, as well as the papers mentioned here, assume that the unobservable factor is i.i.d. (independently, identically distributed). The i.i.d. assumption rules out any effect of business cycle, which is consistent with the "Through The Cycle" rating philosophy, but contradicts the "Point-in-Time' rating philosophy adopted by the banks.

ture ratings at horizons larger than one. If observable factors are included in the probit model, then the rating predictions require the use of an additional dynamic model for forecasting the values of those observable factors at a required number of steps ahead. Thus, no matter what type of factors are considered, the factor dynamics has to be specified and estimated. For the sake of efficiency, recent literature on credit risk shows preference for the use of unobservable factors [10].

# 4 Time-varying Parameter Versus Stochastic Parameter: a Comparison

Comparing the time-varying parameter probit to the stochastic parameter probit in our study is equivalent to comparing the advantages of modeling transition probabilities as $p_{klt} = p_{kl}(\theta_t)$ versus $p_{klt} = p_{kl}(F_t; \tilde{\theta})$. We will show how the assumption of stochastic transition modifies the probabilistic properties of the rating histories. In particular, we will provide insights on the dynamics of transitions under the stochastic specification and the technical details concerning migration correlation.

## 4.1 Deterministic transition matrices

Under the deterministic model considered in Section 2.2:

i) The individual histories are independent,

ii) Any individual history satisfies a Markov process of order 1, with given time dependent parametric transitions.

The prediction of future ratings at all horizons is easy to perform. Let us define the vector of state indicators:

$$Z_{i,t} = (Z_{1,i,t}, \cdots, Z_{K,i,t})', \tag{7}$$

where:

$$Z_{k,i,t} = \begin{cases} 1, & \text{if firm } i \text{ is in rating category } k \text{ at date } t, \\ 0, & \text{otherwise.} \end{cases}$$

[10] The same is true for the credit derivative pricing, where the price of factor volatility is required as an input.

The knowledge of rating histories $(Y_{it})$ is equivalent to the knowledge of indicator histories. Moreover, the prediction of $Z_{i,t+h}$ performed at date $t$ is simply:

$$E[Z_{i,t+h}|Z_{i,t},(P_t)] = P_{t+h-1}P_{t+h-2}\cdots P_t Z_{i,t}. \tag{8}$$

This prediction formula requires the knowledge of future transition matrices.

## 4.2 i.i.d. transition matrices

A simple stochastic transition specification is obtained when the transition matrices ($P_t,\ t = 0, 1, \cdots$) are assumed independent, identically distributed (i.i.d.) in time. In this framework, the elements of the transition matrix are considered as factors $F_t = P_t$, which are i.i.d. variables. We get a model with a quite large number of factors equal to $(K-1)^2$. Let us explain how the assumption of stochastic transition modifies the probabilistic properties of the rating histories.

i) *The individual rating histories become cross-sectionally dependent.*

Indeed, let us consider the covariance between two firm ratings, $Z_{i,t+1}, Z_{j,t+1}$, when their previous ratings are known. By the covariance decomposition equation, we get:

$$Cov[Z_{i,t+1}, Z_{j,t+1}|Z_{i,t}, Z_{j,t}] = Cov[P_t Z_{i,t}, P_t Z_{j,t}|Z_{i,t}, Z_{j,t}].$$

Therefore,

$$Cov[Z_{k,i,t+1}, Z_{k^*,j,t+1}|Z_{l,i,t} = 1, Z_{l^*,j,t} = 1] = Cov(p_{k,l,t}, p_{k^*,l^*,t}), \quad \forall k, k^*, l, l^*.$$

These covariances are generally different from zero. For instance, let us assume current identical ratings $l = l^*$, and consider joint up-grades by one bucket: $k = k^* = l + 1$. We get:

$$Cov[Z_{l+1,i,t+1}, Z_{l+1,j,t+1}|Z_{l,i,t} = 1, Z_{l,j,t} = 1] = Var(p_{l+1,l,t}),$$

which is strictly positive due to the assumption of stochastic transition matrix. This computation shows that stochastic transition matrices yield non-zero cross-sectional correlations.

ii) *Any individual rating history is a homogeneous Markov chain.*

Indeed, from prediction formula (8) and the iterated expectation theorem, it follows that:

$$
\begin{aligned}
& E[Z_{i,t+h}|\underline{Z_{i,t}}] \\
= \; & E[P_{t+h-1}P_{t+h-2}\cdots P_t Z_{i,t}|\underline{Z_{i,t}}] \\
= \; & (EP_t)^h Z_{i,t}, \text{ by the i.i.d. assumption on transition matrices.}
\end{aligned}
$$

Thus, the rating history of a given firm $i$ satisfies a Markov property with a time independent transition matrix, equal to the expected stochastic transition:

$$Q^{(1)} = E(P_t). \tag{9}$$

iii) *Joint analysis of rating histories of two firms.*

The results given above can be extended to a joint analysis of rating histories of two firms $i$ and $j$, say. Typically the bivariate rating process of the two firms is Markov of order 1 with a $(K^2, K^2)$ transition matrix $Q^{(2)}$. The joint transition probabilities are:

$$q^{(2)}_{kk^*ll^*,t} = E[p_{kl,t}\; p_{k^*l^*,t}].$$

In general this matrix is different from the matrix with elements $q^{(1)}_{kl,t}q^{(1)}_{k^*l^*,t}$ as a consequence of migration correlation and cannot be recovered when only $Q^{(1)}$ is known.

The stochastic transition model with iid transition matrices is simple to implement, since, as shown above, it implies that both the univariate and bivariate rating histories are homogenous Markov processes. Equivalently in financial term, it provides a flat term structure of migration correlations and thus flat term structures of spreads of interest rates. This is a drawback of the model with i.i.d. factors [see e.g. Altman (1997) for financial implications]. The factor probit introduced in Section 3.2 does not satisfy the Markov property of individual histories and yields a more flexible term structure.

# 5 Statistical Inference

The parameters of interest in the time-varying probit model can be estimated by maximum likelihood (ML). For the factor model, two estimation procedures are available. These are the simulated maximum likelihood (SML) and the approximated Kalman filter-based maximum likelihood.

## 5.1 Estimation of transition matrices in time-varying parameter probit

When the transition matrices are parameterized by $\theta_t$, the log-likelihood function is:

$$\log L = \sum_{t=1}^{T} \sum_{k=1}^{K} \sum_{l=1}^{K} n_{klt} \log p_{klt}(\theta_t), \tag{10}$$

where $n_{klt}$ denotes the number of firms which migrate from $l$ to $k$ between $t$ and $t+1$ and the identifying constraints on $\theta_t$ are imposed. Thus, $\theta_t = (a^*_{k,t}, m^*_{k,t}, \sigma^*_{k,t},\ \forall k)$.

The expression of the log-likelihood function shows that the set of counts ($n_{klt}$) provides a sufficient statistic for parameters in $\theta_t$. This is due to the cross-sectional homogeneity Assumption A1. It is important to note that the count variables $n_{klt}$ are equal to $n_{klt} = \hat{p}_{klt} n_{l,t}$, where $n_{l,t}$ denotes the number of firms in grade $l$ at the beginning of period $t$. Sample transition matrices $\hat{P}_t$ are available from the rating agencies that report these estimates on a regular basis. We see from the above formula that maximum likelihood estimation also requires data on the structure of the population of firms per rating. Fortunately, the data on numbers of firms in each rating category are reported annually by the rating agencies as well.

The log-likelihood function can be written as follows:

$$\log L = \sum_{t=1}^{T} \sum_{l=1}^{L} n_{lt} \left[ \sum_{k=1}^{K} \hat{p}_{klt} \log p_{klt}(\theta_t) \right].$$

The population of firms changes in time in its size and structure of credit quality rating. The counts $n_{lt}$ allow us to weight in an appropriate way the information available at different dates. The objective criterion comprises a measure $\sum_{k=1}^{K} \hat{p}_{klt} \log p_{klt}(\theta_t)$ for the discrepancy between the sample and the theoretical transition probabilities.

## 5.2 Estimation of stochastic transition matrices

Let us now explain how the maximum likelihood approach is extended to stochastic transition models. Let us consider a factor model, in which the transition probabilities depend on a (multivariate) factor value $F_t$:

$$p_{klt} = p_{kl}(F_t; \tilde{\theta}), \quad \text{say}, \tag{11}$$

and the factor satisfies a Gaussian VAR model:

$$F_t = AF_{t-1} + \eta_t, \tag{12}$$

where the errors $\eta_t$ are i.i.d. standard normal vectors.

The parameters to be estimated are (i) the parameters $\tilde{\theta}$ defining the transition probabilities, (ii) the parameter (matrix of parameters) $A$ which characterizes the factor dynamics.

The maximum likelihood method cannot be used for estimation of the factor probit model due to the complicated form of the likelihood function, derived below. If both rating and factor histories were observed, the likelihood function would be:

$$L^*(\tilde{\theta}, A) = \prod_{t=1}^{T}\prod_{k=1}^{K}\prod_{l=1}^{K}(p_{kl}(F_t;\tilde{\theta}))^{n_{klt}}\Psi(f_1, ..., f_T; A), \tag{13}$$

where $\Psi$ denotes the joint density of the factor values $F_1, \cdots, F_T$. When the factors are not observed, the formula given above has to be integrated with respect to $F_1, \cdots, F_T$. We get a likelihood function based on the rating histories only (with counts $n_{klt}$ as summary statictics):

$$L(\tilde{\theta}, A) = \int \cdots \int \left\{ \prod_{t=1}^{T}\prod_{k=1}^{K}\prod_{l=1}^{K}(p_{kl}(F_t;\tilde{\theta}))^{n_{klt}}\Psi(f_1, \cdots, f_T; A)df_1 \cdots df_T \right\}.$$

The log-likelihood function: involves a multivariate integral of dimension equal to $MT$, which can be very large.

#### 5.2.1 Simulated maximum likelihood

In the simulated maximum likelihood approach, the multivariate integral is replaced by an approximation obtained by simulations [see e.g. Gouriéroux and Monfort (1995) for a survey]. The simulated maximum likelihood (SML) estimator is defined as:

$$(\hat{\tilde{\theta}}, \hat{A}) = \underset{\tilde{\theta}, A}{Argmax} \log \left\{ \frac{1}{S}\sum_{s=1}^{S}\left\{ \prod_{t=1}^{T}\prod_{k=1}^{K}\prod_{l=1}^{K}(p_{kl}(f_t^s(A);\tilde{\theta}))^{n_{klt}} \right\} \right\},$$

where $S$ denotes the number of replications and, for each $s$, the simulated factor values are computed recursively by:

$$f_t^s(A) = Af_{t-1}^s(A) + \eta_t^s, \;\; t = -H, \cdots, T, \tag{14}$$

with the initial condition $f^s_{-H}(A) = 0$, and errors $\eta^s_t$ independently drawn in the multivariate standard normal. The initial condition is fixed at a past date to ensure stationary behaviour of the factor in the period of interest starting at $t = 1$.

### 5.2.2 Approximation by a linear factor model

Although the SML estimators are consistent and efficient for large $S$, practitioners may find the procedure quite difficult and time consuming. Therefore, we present another estimation method providing consistent and fully efficient estimators when the cross-sectional dimension, that is, the number of firms in the sample is large [Gourieroux, Monfort (2004)].

Let us consider the factor ordered qualitative model with latent means driven by Gaussian autoregressive factors [see equations (4-6)]. If numbers of firms per rating category $n_{lt}$, are sufficiently large, the estimators $\hat{m}_{l,t}$ of latent means, computed per year, are close to the true latent means. Thus we can write:

$$\begin{aligned} \hat{m}_{l,t} &\approx \sum_{i=1}^{M} \gamma_{l,i} F_{i,t} + \delta_l + \nu_{l,t}, \\ F_{t+1} &= A F_t + \eta_t, \end{aligned} \tag{15}$$

where the errors ($\nu_{l,t}$) are Gaussian by the Central Limit Theorem applied to the estimator $\hat{m}_{l,t}$. We get an approximately Gaussian linear factor model, which can be estimated by means of the standard Kalman filter. It provides estimates of parameters $\gamma, \delta,$ A and $\Omega = Var(\nu)$, and also the approximated factor path (filtering).

If the number of firms is not sufficiently large, the method will loose its optimality. However, it can still be used as the first step estimation in a preliminary analysis of the stochastic migration model, in particular for determining the number of factors and constraining their dynamics. In the second step, estimators obtained from the first step can be used as initial values in the numerical algorithm to maximize the simulated likelihood function.

# 6 Application

The deterministic and stochastic transition ordered qualitative models are estimated on aggregate data provided by Standard & Poor's [see Brady,Bos(2002), Brady, Vazza, Bos (2003)]. In the first section, we describe the data set and explain how to correct for bias due to missing data on non-rated companies. In Section 6.2, the time varying parameter probit model is estimated. The estimation is carried out independently for each year, due to the additive form of the log-likelihood function. It allows us to find the estimated thresholds $a^*_{k,t}, k = 1, \cdots, (K-1)$, as well as the estimated means $m^*_{l,t}, l = 1, \cdots, (K-1)$ and variances $\sigma^{*2}_{l,t}, l = 1, \cdots, (K-1)$ per rating category (under the identifying constraints $a^*_{1,t} = 0, \sigma^*_{1,t} = 1,\ \forall t$), and to observe how they vary in time. Various tests for time stability of the parameters are also performed. In Section 6.3, we focus on the model with fixed thresholds: $a_{k,t} = a_k$ and variances $\sigma_{l,t} = \sigma_l$ constant in time, and on the time series properties of the conditional means. This is a preliminary step before estimating a factor probit model with serial correlation in Section 6.4.

## 6.1 Data set

The data set used in this paper comes from "Rating Performance 2002" provided by Standard & Poor's, available at "www.standardandpoors.com". The data consists of annual transition matrices from 1981 to 2002, reported in "Static Pool One-Year Transition Matrices" by Standard & Poor's (2003) [11].

According to S&P rating system, there are $8$ rating categories. These are "AAA","AA", "A", "BBB", "BB", "B", "CCC" and "D" ranked from the lowest up to the highest risk [for definition of ratings and comparison of the rating systems, see Foulcher et al. (2004)]. Since the published ratings focus on individual bonds, S&P "convert their bond rating to issuer ratings by considering the implied long-term senior unsecured rating" [see Bangia et alii (2002)][12]. For computational convenience, we use quantitative indicators (dummies) "1","2",..., "8" for the $K = 8$ ratings: "1" stands for the highest rating category "AAA", and "8" for default "D". The following scheme illustrates the approach:

[11] See Brady, Vazza and Bos (2003) for the precise definition of the so-called static pool.

[12] Similarly Nickell, Perraudin and Varotto (2000) considered long term corporate and sovereign bond ratings on the Moody's data base.

[insert Table 1: Rating scheme]

The transition matrix displays all rating movements during a one year period and accounts for missing data . A typical example of transition matrix for year 1997 is given in Table 2.

[insert Table 2: Number of issuers and transition matrix for year 1997 in %]

Table 2 has $10$ columns and $7$ rows. The rows represent the rating categories from which transitions are to be made: the first row refers to rating category "1", i.e. "AAA", and the last row refers to rating category "7", i.e. "CCC". The category "8" for "D" is excluded from the rows because once a firm defaults, it remains in default forever. The first column, named "Issuers", provides the numbers of long-term rated issuers per rating category on January 1st, 1997, that is $n_{lt}$ at the beginning of the period. The columns 2 to 9 correspond to rating categories "1" to "8" to which transitions will be made until the end of year 1997. The last column "9" corresponds to the alternative "N.R.", which pertains to issuers who were not rated at the end of year, but were rated at the beginning of the year. As pointed out by Brady, Vazza and Bos (2003), *Ratings are withdrawn when an entity's entire debt is paid off or when the program or programs rated are terminated and the relevant debt extinguished. They may also occur as a result of mergers and acquisitions. Others are withdrawn because of a lack of cooperation*. From the statistical point of view, the rating cannot be assigned due to a lack of information concerning the balance sheet of firms. Thus these data are missing. The proportions of missing data in the available data bases from S&P and Moody's are rather high (between 10% and 20%).

Let us now discuss more precisely the data in columns 1 to 9. These are the observed transition frequencies for year 1997, including the "N.R." alternative. For example, the third row shows that out of "1161" firms rated "A" at the beginning of the year 1997: no one were rated as "AAA" at the end of the year 1997; 1.64% were upgraded to "AA"; 89.15% stayed in the same category; the proportion downrated to "BBB", "BB","B" were 3.7%, 0.17% and 0.43%, respectively. The last number, 4.91%, stands for the proportion of nonrated firms.

In Table 2, the transition probabilities for nonrated firms (N.R. henceforth) at the beginning of the year are not provided. Two approaches can be followed:

(i) adding the category "N.R." to the state space;

(ii) retaining only the ratings “1” to “8”, i.e. “AAA” to “D”.

The first approach requires including an additional row for companies, which are not rated at the beginning of 1997. Generally this information is not provided by the rating agencies, likely for confidentiality reasons. Indeed, such information could reveal the changes in the population of firms which request a rating from the rating agency, and their credit quality. Because approach (i) would require additional information about firms, which is not easily available, we follow approach (ii) which consists in excluding the “N.R.” alternative from the transition matrix, as in Nickell et al. (2000), Bangia et al. (2002) and Foulcher et. al. (2006). For this purpose, the incomplete transition matrix by S&P is adjusted, by allocating proportionally the weights of “N.R.” firms among the other categories [13]. We get the so-called “N.R.-adjusted” transition matrix of year 1997.

Let us consider the third row of matrix in Table 2, for example. The N.R.-adjusted” transition frequency from “A” to “AA”, is computed from the ratio $1.64/(1 - 0.0491)$ and equals 1.72%. The “N.R.-adjusted” transition matrices are used in our analysis as measures of unconstrained transition frequencies $p_{k,l,t}, \quad k, l = 1, \cdots, 8$.

All matrices in our study display the typical pattern of rating transitions documented in the literature. The frequencies on the main diagonal are close to one. This suggests that changes in rating categories don’t happen very often in a one-year span. The transition frequencies on the two diagonals below and above the main one are significantly different from zero, while the remaining elements of the matrix are close or equal to zero. The zero entries indicate that the recorded changes in ratings (down- or up-grades) are at most by one bucket. In fact, a one year sampling period may be too short to observe a significant number of rating transitions by more than one bucket. Such rating transitions are observed when unpredicted corporate failures occur. For these firms, the rating agencies quickly perform several down-grades to correct for the prediction error. This effect can be seen in the bottom right corner of the matrix in Table 2. We also observe more uncertainty about the future ratings of low rated companies than of the high rated ones.

[13] This assignment of “N.R.” companies assumes no selection bias.

## 6.2 Estimation of time varying parameter probit models

In the first step, we consider estimation of a sequence of time varying parameter probit models on the "N.R.-adjusted" transition matrices computed from the S&P data set. The estimation, performed separately for each year due to the additive form of the log-likelihood function, provides approximation of the thresholds $a^*_{k,t}$, latent means $m^*_{l,t}$ and latent standard errors $\sigma^*_{l,t}$. Recall, that for identification, we constrain $a^*_{1,t} = 0, \sigma^*_{1,t} = 1,\ \forall t$, that is assume zero threshold and variance 1 for the best credit rating category 1 (resp. AAA) in the whole sampling period.

The estimation method is the maximum likelihood (ML) approach, for which the criterion function corresponds to the Kullback-Leibler information criterion (KLIC):

$$\sum_{k=1}^{K} \sum_{l=1}^{K} \hat{p}_{k,l,t} \log p_{k,l,t}(\theta_t).$$

Let us discuss in detail the estimated parameters for all risk rating categories, at one selected point of time. By definition for a given year $t$, the thresholds $a_{k,t}$ increase with risk, i.e. $k$, and the same is true for the transformed thresholds $a^*_{k,t}$. This is what we observe in Table a.1 when we look at the estimated $\hat{a}^*_{k,t}$ reported in rows which pertain to different years $t$ of the sampling period. Similarly, we expect the means of the score for different rating categories to increase with risk in each year. This property is revealed by the data when we look at the estimates in each row of Table a.2.

The estimated transformed standard deviations $\hat{\sigma}^*_{l,t}$ are given in Table a.3. The ordering of $\sigma_{l,t}$ and $\sigma^*_{l,t}$ is identical for any given year $t$. However, contrary to $a^*$ and $m^*$, the standard deviations don't increase with risk in each year (row). This allows us to find out which among the credit quality rating categories are the most heterogeneous ones, i.e. feature the highest dispersion of score values. For all years in the sampling period (i.e. in all rows) we find the highest heterogeneity in category B, followed by AAA. The most homogeneous class is A, followed by CCC.

The differences in levels of residual heterogeneity accross rating categories are important for computing the CreditVaR. Indeed, the CreditVaR is a quantile of the risk quality distribution, and is very sensitive to the dispersion of firms in each rating category. It comes as no surprise that the highest heterogeneity is found in firms rated B. Rating BB is the last one for firms to be qualified as investment bonds issuers. The bonds of firms rated less than B are considered speculative, and therefore much more risky. Since the decision to upgrade a firm to the group of investment bonds has important economic implications, the rating companies delay this decision until they collect a sufficient amount of strong evidence. This can explain the large heterogeneity in rating B.

So far we compared the estimated parameters accross rating categories, separately in each row. One may be interested in the evolution of parameter estimates in time, that is, in comparing the parameter values in different rows. The direct comparison of parameter values is misleading, because for example:

$$a^*_{k,1992}/a^*_{k,1991} = \frac{a_{k,1992} - a_{1,1992}}{\sigma_{1,1992}} \frac{\sigma_{1,1991}}{a_{k,1991} - a_{1,1991}},$$

differs form $a_{k,1992}/a_{k,1991}$.

Nevertheless some dynamic properties of the initial parameters of the model can be investigated. In years $1981 - 1987$, the estimates seem erratic while in the following years they look almost stable in time. It seems that this pattern is not due to changes in credit worthiness of corporates, but rather to different data collecting techniques. Indeed, the quality of data is poor at the beginning of the sampling period while it improves for more recent years. In particular, the data base for the years 1981-1987 is currently under revision by S&P. It occurs that transition matrices for years 1981-1987 reported in 2002 are very different from those provided in 2003. Moreover, the structure of the population of firms by geographical region (North America, Western Europe, Asia) and industry (Manufacturing, Utilities, Financial, etc.) has been more stable after 1990, as shown in [Bangia et alii (2002), Figure 5, or in Nickell et alii(2000) for Moody's data set]. For estimation of a stochastic factor model, the use of a data base including the first years of the sampling period, during which the population of foreign firms and firms from the non- manufacturing sector is lower, seems inappropriate. The estimated factor would likely capture the structural change of the data base instead of measuring the risk fundamental. For this reason and for quality of input, we keep for further analysis only data from 1990 to 2002.

## 6.3 Estimation of the constrained time varying parameter probit

Prior to introducing stochastic factors, we investigate if some among the transformed parameters can be assumed constant in the period 1990-2002. For this purpose, we reestimate the time-varying probit models under the following intertemporal constraints: i) constant transformed thresholds, ii) constant transformed thresholds and constant transformed latent variances, and iii) constant transformed thresholds and constant transformed latent means. The estimation of each restricted model is performed in one step from the whole sample, and not year by year as for the unconstrained model, due to the intertemporal constraint imposed.

We find that the model in which both transformed thresholds and latent variances are constant in time is quite close to the unconstrained model in terms of goodness of fit, consistently throughout the sampling period. We also find that the hypothesis of constant transformed thresholds and means is rejected by the likelihood ratio tests [14].

The evidence in favor of constant transformed thresholds suggests that S&P hasn't significantly modified the definition of the rating categories AAA,...,CCC in the period 1990-2002. This argument seems plausible. Therefore, from now on, we consider time invariant thresholds and latent variances.

The estimators of thresholds and latent variances are given in Table 3 under the identifying constraint $a_1^* = 0, \sigma_1^* = 1$.

[Insert Table 6: Estimated thresholds and variances in the constrained probit]

The estimates of time varying transformed latent means from the constrained model are reported in Table a.4 in Appendix 1. These estimates will be used as inputs in estimation of the factor probit model.

## 6.4 Factor models

Let us now consider serially dependent stochastic transition matrices, modelled by factor probit with stochastic latent means.

[14] The degrees of freedom for the test statistic are equal to (12-1)(6+7)=143, given that we have the sampling period of length 12, 6 threshold parameters and 7 mean parameters.

### 6.4.1 The factor probit model

To define the factor probit model, we consider a time varying parameter probit model with parameters written as functions of unobservable factors. However, according to the results reported in Section 6.3, there is no evidence for time variation in thresholds and latent variances. Therefore, we specify a probit model with constant thresholds $a_{k,t} = a_k$, constant variances $\sigma^2_{l,t} = \sigma^2_l$, and time varying latent means driven by a finite number of factors as in equations (5)-(6).

In the factor probit model, some identifying restrictions are also necessary. As before, we impose $a^*_1 = 0, \sigma^*_1 = 1$. Under this constraint, the transformed latent means $m^*_{l,t}$ are affine functions of the initial latent means $m_{l,t}$. It is easy to see that it is equivalent to write a factor representation for $m_{l,t},\ l = 1, ..., K$ or for $m^*_{l,t},\ l = 1, ..., K$:

$$m^*_{l,t} = \sum_{i=1}^{M} \gamma^*_{l,i} F_{i,t} + \delta^*_l, \quad \text{say.}$$

Additional identifying restrictions are required to avoid the multiplicity of factor representations, obtained by replacing the factor by its (invertible) affine transformation. Therefore, we also impose $V\eta_t = Id$, that is assume that the variance of factor innovation in equation (6) is an identity matrix.

The consistent estimators of latent means $\hat{m}^*_{l,t},\ \forall l, t$ obtained from the constrained time varying parameter model discussed in the last section are used to investigate the existence of a factor representation and to estimate the number of underlying factors. For this purpose, we perform a principal component analysis based on the series of estimated latent means. The spectral decomposition of the sample variance-covariance matrix of the series $(m^*_{l,t}), l = 1, ..., K$ give the following eigenvalues: 1.584e-01, 4.942e-03, 1.487e-03, 8.490e-04, 3.082e-04,... The gap between the largest eigenvalue that is significantly greater than the remaining ones, indicates a one factor model. This result suggests that only one factor drives the latent means in time.

### 6.4.2 Linear approximation of the factor model

Let us now consider the one factor model:

$$m^*_{l,t} = \gamma^*_l f_t + \delta^*_l,$$
$$f_t = \rho f_{t-1} + \eta_t,$$

where $\{\eta_t\}$ are independent standard Gaussian variables, and $\gamma^*_l, \delta^*_l$ are deterministic coefficients. The $\delta^*_l$ parameter provides a measure of expected risk averaged over time (since $Ef_t = 0$), whereas the dynamic effect is captured by the sensitivity coefficient $\gamma^*_l$ on factor $f_t$. For a factor which represents aggregate risk, a large value of coefficient $\gamma^*_l$ implies high risk sensitivity of the score means. Note that the last identification restriction consists in choosing a factor which increases with aggregate risk, that is, in fixing the sign in order to get positive sensitivity coefficients.

Since the transformed latent means can be well approximated by their cross-sectional estimates $\hat{m}^*_{l,t}$, we first consider the approximate model:

$$\hat{m}^*_{l,t} = \gamma^*_l f_t + \delta^*_l + \nu_{l,t},$$
$$f_t = \rho f_{t-1} + \eta_t$$

where $\{\eta_t\}$ are $IIN(0,1)$, and the measurement errors $\nu_{l,t}$ are approximately Gaussian [15]. The advantage of this specification is that it is a special case of a linear factor model, for which software is available.

Prior to applying the linear Kalman filter to the approximate model [16], a test of stationarity is required. We found no evidence indicating the presence of a unit root in the dynamics of latent means (the highest first-order correlation was detected in rating categories B and BB, equal to 0.5 and 0.6, respectively). The estimation of the approximated latent one-factor model is reported in Table 4.

[Insert Table 4: Estimation of the approximate factor model]

[15]As already mentioned, the error terms $\nu_{l,t}$ are due to the estimation error. They are approximately Gaussian by Central Limit Theorem. They are also heteroscedastic. The heteroscedasticity is not completely taken into account in this first step, providing inefficient, but consistent estimators.

[16]It is shown in Gourieroux, Monfort (2004) that this approach provides consistent estimators. The estimators of $\gamma^*_l, \delta^*_l$ converge at speed $(\sqrt{nT})^{-1}$, where $n$ denotes the cross-sectional dimension, whereas the estimator of $\rho$ converges at rate $(\sqrt{T})^{-1}$. Moreover, the latter one is asymptotically efficient when $T \to \infty$.

The identification restriction imposed in our estimation procedure provides a factor which increases with risk. The larger the factor value, the larger is the expected score, that is, the expected probability of default. As expected the sensitivity coefficients are all positive and tend to be much larger in the risky rating categories.

Let us now study the dispersion of latent scores in each rating category. The dispersion can be measured in different ways depending on the amount of information available. If the factor $f_t$ is not known, the dispersion is measured by the marginal variance of $S_{l,t}$ given by:

$$Var(S_{lt}) = \gamma_l^{*2} Var(f_t) + \sigma_l^{*2} = \frac{\gamma_l^{*2}}{1-\rho^2} + \sigma_l^{*2}.$$

When the factor is known, the conditional variance $\sigma_l^{*2}$ can be used to measure the dispersion. In Section 6.2, the discussion of score dispersion (heterogeneity within rating categories) was based on the time varying parameter probit model, and concerned the conditional variance of the score given a time varying mean, now approximated by the factor. We observed that the conditional variance was large in rating category AAA and small in rating category CCC. The outcomes are different when marginal variances are considered. We see that the sensitivity coefficient $\gamma_l^*$ is close to zero for rating category AAA, leading to marginal variance close to 1. In contrast, the sensitivity coefficient $\gamma_l^* = 5.8$ for rating category CCC, yields marginal variance equal to $\frac{5.8^2}{1-(0.02)^2} + (0.218)^2$, which is much larger than the marginal variance for rating category AAA.

The filtered factor values recovered from the Kalman filter [17] for the approximate factor model are given in Table 5.

[Insert Table 5: Filtered factor values]

The factor dynamics is similar to the evolution of the total default rate as reported for instance in Brady, Bos(2002), Chart 5, or in Hamilton, Cantor and Ou (2002), Exhibit 4. Interpretation of the factor will be given in the next section.

[17] It can be shown that the filtered factor values converge to the true factor values at speed $(1/\sqrt{n})$ [see Gourieroux, Monfort (2004)].

#### 6.4.3 Simulated maximum likelihood for the factor probit model

In this section, the one-factor probit model is estimated by simulated maximum likelihood. This method is (asymptotically) efficient and could produce more accurate results in finite sample than the estimation of the linear approximation of the model. The estimates are given in Table 6.

[Insert Table 6: SML estimation of one factor probit]

In the simulation procedure, 2000 replications were used. The estimated $\gamma^*$ and $\delta^*$ coefficients are quite close to those obtained from the linear approximation. The estimates of $\sigma^*$, $\rho$ and thresholds differ, which reveals the lack of robustness in heterogeneity estimation. Note however that the estimated serial correlation $\rho$ is smaller which can have a significant impact on predictions.

Let us briefly comment on the value of the factor autoregressive coefficient $\rho$. Since the estimated value is small, one may question the presence of serial dependence and prefer to use an iid factor instead. The final argument in favor or against serial dependence in the factor can only be provided once the predictions of future ratings at various horizons are computed and compared. Indeed, the crucial role of the factor autoregressive coefficient is in determining the pattern of the term structure of future rating. In Section 4.2 we noted that, in the absence of serial correlation, the term structure of rating is flat. We will see later in the text that the empirical results indicate that this is not the case, and that a small value of the autoregressive coefficients can have a strong impact on the term structure of future ratings because the nonlinear features of the model amplify its effect.

To facilitate interpretation of score dispersion in each rating category, we give in Table 7 the marginal variance of the quantitative score, along with the proportion of variance explained by the factor.

[Insert Table 7: Score dispersion]

More accurate filtered latent factor values can be derived by exploiting the large cross sectional dimension, which allows to avoid the complicated nonlinear filtering. For $\hat{\gamma}_l^*$, $\hat{\delta}_l^*$ denoting the coefficient estimates of Table 6 and the estimated mean $\hat{m}_{lt}^*$ given in Table a.4 in Appendix 1, we have approximately:

$$\hat{m}^*_{lt} = \hat{\gamma}^*_l f_t + \hat{\delta}^*_l \quad l = 2, \cdots, 7, \ \ \forall t.$$

The filtered value of $f_t$, which takes into account the improved estimates, is the OLS estimator obtained by regressing $\hat{m}^*_{lt} - \hat{\delta}^*_l$ on $\hat{\gamma}^*_l$ for each time $t$:

$$\tilde{f}_t = \frac{\sum_{l=2}^{7} (\hat{m}^*_{lt} - \hat{\delta}^*_l)\hat{\gamma}^*_l}{\sum_{l=2}^{7} \hat{\gamma}^{*2}_l}.$$

The estimated values $\tilde{f}_t, t = 1, ..., T$ have to be demeaned and standardized to satisfy the factor identification restriction. The standardization is performed by dividing by the standard deviation of $\tilde{f}_t - E(\tilde{f}_t) - \rho(\tilde{f}_{t-1} - E(\tilde{f}_t))$. The smoothed factor values $\tilde{f}_t, t = 1, ..., T$ are given in Table 11 and plotted in Figure 1.

[Insert Table 8: Filtered factor values for factor probit]

[Insert Figure 1: Filtered latent factor for factor probit model]

It is natural to compare the factor dynamics with a proxy of the general state of economy. In the literature, a commonly used proxy for the general state of economy is the indicator of recessions and expansions [see e.g. Rosch (2003), Loffler (2004)]. For example, this approach is adopted in Bangia et al. (2002), where the NBER indicator is used as a factor. Such a factor admits only two values corresponding to each of the states. Our findings suggest that the underlying factor is continuously valued [18] and does not show a two regime dynamics (This is an example of misspecifications due to the use of ad hoc observable factors). In fact, the US economy was in expansion for most of the sampling period, which corresponds to a hump shaped growth rate. We observe a U-shaped pattern of the factor, indicating negative dependence between growth rate and risk.

Given the estimates in Table 2 and the smoothed factor values, we can compare the time varying means $\hat{m}^*_{l,t}$ estimated from the time varying parameter probit with the fitted values $\hat{\gamma}^*_{l,t}\hat{f}_t + \hat{\delta}^*_l$ based on the factor probit. This comparison allows us to verify whether the one factor probit is capable of reproducing the patterns of expected scores in various

[18] Some authors felt they had to increase the number of states to "cycle through", "cycle normal", "cycle peak" [see Nickell et al. (2000)].

rating categories. To do this, we plot in Figure 2 the path of score means estimated from the time varying parameter probit and from the factor probit for each credit rating category.

[Insert Figure 2: Means and Fitted Means]

The one factor model is successful in reproducing the general pattern of expected risk in each rating class. However, a second factor would be required to capture the pronounced through in 1992-93 observed in the risky rating categories.

# 7 Prediction of future ratings

In the banking sector, the rating transitions models are used to predict the future structure of risk on a given credit portfolio, in the general framework of risk management and control. The rating prediction is of particular importance for computing the minimum required capital based on the CreditVaR, and for the assessment of future risk exposure.

The advantages of the unobservable factor ordered probit model is the possibility of making predictions, and examining correlation in credit migrations. In particular, since the model does not contain any observable explanatory variable, we don't need to predict future values of variables (such as the business cycle, unemployment or riskfree interest rate) in order to predict future ratings. We first recall the prediction formulas and next compute the predictions for the S&P data set.

## 7.1 Rating prediction for one firm

Let us first consider a given firm. If the future values of the factor $f_{T+1}, \cdots, f_{T+H}$ were known, the transition matrix at horizon $H$ would be:

$$P(H; f_{T+1}, \cdots, f_{T+H}) = P(f_{T+H}) \cdots P(f_{T+1}),$$

and the distribution of its rating at date $T + H$ when its rating at date $T$ is $l$ would correspond to the $l^{th}$ row of matrix $P(H; f_{T+1}, \cdots, f_{T+H})$.

When the future factor values are not observed, the matrix above becomes stochastic, and has to be integrated with respect to $f_{T+1}, \cdots, f_{T+H}$ conditional on $f_T$. The integrated matrix is:

$$Q^{(1)}(H|f_T) = E[P(f_{T+H}) \cdots P(f_{T+1})|f_T].$$

This matrix has no closed form expression, but the prediction of the rating at $T + H$ can be obtained by simulation along the following steps:

Step 1: Simulate a path of the noise $\eta^s_{T+1}, \cdots, \eta^s_{T+H}$, and compute the associated future factor values $f^s_{T+1}, \cdots, f^s_{T+H}$ from the autoregressive factor formula;

Step 2: Compute the matrix $P(H; f^s_{T+1}, \cdots, f^s_{T+H})$ and its row number $l$: $P^l(H; f^s_{T+1}, \cdots, f^s_{T+H})$, say.

Step 3: Replicate the simulation $S$ times and compute $\frac{1}{S}\sum_{s=1}^{S} P^l(H; f^s_{T+1}, \cdots, f^s_{T+H})$, which is an approximation of the $l^{th}$ row of the matrix $Q^{(1)}(H|f_T)$.

## 7.2 Rating prediction for two firms

Let us now consider a credit portfolio for two firms $i$ and $j$, say, whose ratings are known at date $T$. Their joint transition is summarized by a matrix $(K^2, K^2)$, which gives the probability that firms $i$ and $j$ currently in rating category $l_i$ and $l_j$, respectively, are in category $k_i$ and $k_j$ at $T + 1$, say. If the future value of the factor is known, this joint transition matrix is:

$$P^{(2)}(f_{T+1}) = P(f_{T+1}) \otimes P(f_{T+1}),$$

where $\otimes$ denotes the tensor product, which associates with the matrices $A, B$, the matrix $A \otimes B$ with block elements $(a_{ij}B)$. If the sequence of future values is known, the joint migration matrix at horizon $H$ becomes:

$$P^{(2)}(H; f_{T+1} \cdots f_{T+H}) = P^{(2)}(f_{T+1}) \cdots P^{(2)}(f_{T+H}).$$

When only the current factor value is known, this matrix has to be integrated to get:

$$\begin{aligned} Q^{(2)}(H|f_T) &= E[P^{(2)}(f_{T+1}) \cdots P^{(2)}(f_{T+H})|f_T] \\ &= E[P(f_{T+H}) \otimes P(f_{T+H}) \cdots P(f_{T+1}) \otimes P(f_{T+1})|f_T]. \end{aligned}$$

This matrix can not be written in general as a tensor product, which means that migration correlations have been created by the common effect of the unobservable factor.

Again this matrix has no closed form expression, but can be well approximated by simulation.

The approach described above is an example that can be easily extended to a portfolio of credits for more than two firms.

## 7.3 Prediction of future ratings- empirical results

Let us now consider the prediction of future ratings from the S&P data base. For this purpose the factor value is fixed at its filtered value computed for 2002, that is $f_T = 1.607$, and the parameters at their estimates given in Table 6.

We perform $S = 5000$ replications of a simulated factor path to get the matrices $Q^{(1)}(H|f_T)$, $Q^{(2)}(H|f_T)$ at horizon 1 year, 2 year, 5 year and 10 year. The matrices $Q^{(1)}$ are given in Table a.5, Appendix 1. We also compute the difference $Q^{(2)}(H|f_T) - Q^{(1)}(H|f_T) \otimes Q^{(1)}(H|f_T)$ for the following joint migrations corresponding to joint upgrades $(k,h) \dashrightarrow (k-1,h-1)$, to joint stability $(k,h) \dashrightarrow (k,h)$ and to joint downgrades $(k,h) \dashrightarrow (k+1,h+1)$. (see Tables a.6 - a.9, Appendix 1) to get more information about the term structure of the migration correlations.

## 7.4 Migration Correlation

The matrices $Q^{(1)}$ and $Q^{(2)}$ can be used to compute the migration correlations for joint transitions of two firms. The firms don't need to start from the same rating and end up in the same rating as well [19]. Also, the migration may take one , two or more years. Therefore, the correlations are computed at different horizons. More precisely, let us consider two firms i and j, who are currently in rating categories $l_i$ and $l_j$ , respectively. Let us also define the indicator function:

$$
\begin{aligned}
Z_{k_i,T+H} &= 1, \quad \text{if firm i is in rating } k_i \text{ at } T+H \text{ ,} \\
&= 0, \quad \text{otherwise,} \\
Z_{k_j,T+H} &= 1, \quad \text{if firm j is in rating } k_j \text{ at } T+H \text{ ,} \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

[19] The Basle Committee suggests to group borrowers into homogeneous risk buckets, defined by credit quality rating, country and industrial sector, and to examine the risk correlation within buckets [see also Gordy (2003), Rosch (2005)]. Implicitly, this approach assumes all migration correlations equal to zero for firms which do not start migrating from the same rating category (or don't end up in the same rating category).

The migration correlation for the transition $l_i \to k_i$, $l_j \to k_j$ at horizon $H$ is defined as $Corr(Z_{k_i,T+H}, Z_{k_j,T+H}|I_t)$. The migration correlation depends on the type of transition considered, on the horizon and on the available information.

If the information set includes the value of the factor at time T, then the correlation is given by the following formula:

$$\begin{aligned} & Corr(Z_{k_i,T+H}, Z_{k_j,T+H}|l_i, l_j, f_T) \\ = & \frac{((Q^{(2)}[H|f_T] - Q^{(1)}[H|f_T])Q^{(1)}[H|f_T])_{(l_i,k_i),(l_j,k_j)}}{\sqrt{Q^{(1)}[H|f_T]_{l_i,k_i}[1 - Q^{(1)}[H|f_T]]_{l_i,k_i}}\sqrt{Q^{(1)}[H|f_T]_{l_j,k_j}[1 - Q^{(1)}[H|f_T]]_{l_j,k_j}}} \end{aligned}$$

If the future factor path were completely observed, the migration correlation would be zero. Since the factor is assumed observed up to time T only, we can expect that $f_{T+1}, ..., f_{T+H}$ have an impact on the latent scores resulting in migration correlation. In Table 9 we give some values of migration correlation computed for $T = 2002$ and with $f_T = f_{2002}$ set equal to the filtered value from the factor probit.

[Insert Table 9: Term structure of correlations for migration BB $\to$ B, 2 firms]

The first row of Table 9 displays migration correlations for two firms who make a transition from rating BB to rating B, in one, two, five and 10 years. We observe negative correlations at short horizons and we find that the magnitude of correlations depends on the horizon. This points to the existence of a term structure of migration correlations which should be taken into account when the term structure of spreads for credit derivatives written on several firms is examined. We observe that the term structure of migration correlations depends on the initial and final ratings. Also, contrary to a stylized fact established in earlier literature [see Turnbull (2004)], the default correlation does not necessarily increase with the horizon. In the second part of Table 9, we show the correlations for joint absence of migration at the horizon of 1 year. These correlations are, in general, very small except for the risky firms.

The total number of correlations computable from the matrices provided in our study is close to 8200. More precisely, the exact number of correlations is $K^2(K^2+1)/2 \times 4$, where 4 is the number of horizons considered in our analysis. The migration correlations, including default correlations, are generally quite small. In particular their values are much smaller than those announced by the Basle Committee [20]. Moreover, the migration correlations depend on: 1) the horizon, 2) the rating categories in which the migration is initiated, and 3) the rating categories in which the migration ends. These three features, disregarded by the Basle Committee, seem to have a crucial impact on capital charge.

# 8 Concluding remarks

The banking supervisory authorities advocate the use of a factor probit model for predicting future risk on a credit portfolio. The official documents provide however no details concerning the specification of the factor variable. While most applied researchers interpret the factor as an observed macroeconomic variable, our approach is based on an unobserved (latent) driving process. An advantage of a latent factor model is that it allows for computation of migration correlations (default correlations). Additionally, by introducing serial correlation of the latent factor one can i) recover the intrinsic driving variable(s) which determine the dynamics of credit ratings, ii) predict future ratings for a portfolio of firms at various horizons, and compute the CreditVaR, iii) calculate the migration correlations and their term structure. In these regards, the choice of the dynamic latent factor model complies with the following suggestion made by Couderc, Renault (2004): ”Instead of attempting to continuously improve the selection of macro-factors proxying the health of industrial sectors, more predictive proxies could be endogeneously extracted from the (migration) process itself.”

Our empirical results contribute to the on-going discussion concerning the two alternative credit rating philosophies, which are ”point-in-time’ (PIT) and ”through-the-cycle” (TTC). By introducing a dynamic latent factor we are able to account for the business cycle effect in rating transitions. In contrast, an iid latent factor would be consistent with the TTC approach, recently criticized in a number of papers [see e.g. Basle Committee on Banking Supervision (2000), Treacy, Carey (2000), Carey, Hrycay (2001), Crouhy, Galay, Mark (2001)]. The gap between the PIT and TTC interpretations explains in part the numerical differences between the values of estimated migration correlations based on our model and those fixed by the Basle Committee. The TTC approach disregards the busi-

[20] Similar results were obtained in other recent applied studies [see e.g. Gordy (2000), Foulcher, Gourieroux, Tiomo (2004), Gagliardini, Gourieroux (2005a) Figures 4 and 5, Rosch (2005) Tables 4 and 5].

ness cycle effect, and provides higher values of cross-sectional correlations than the PIT. This doesn't however fully explain why the differences are so large. The credit ratings provided by S&P have been officially declared compatible with the TTC methodology [see e.g. the discussion in Couderc, Renault (2004)]. Our latent factor model shows that, contrary to this claim, the business cycle plays a very important role in the S&P ratings.

Table 1: Rating scheme

| R.C | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| | AAA | AA | A | BBB | BB | B | CCC |
| TH | $< a_1$ | $a_1 \leq \cdots < a_2$ | $a_2 \leq \cdots < a_3$ | $a_3 \leq \cdots < a_4$ | $a_4 \leq \cdots < a_5$ | $a_5 \leq \cdots \leq a_6$ | $a_6 \leq \cdots < a_7$ |

Note: R.C. and TH. stand for "Rating category" and "threshold" respectively.

Table 2: Number of issuers and transition matrix for year 1997 in %

| | Issuers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 199 | 94.47 | 4.02 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.51 |
| 2 | 586 | 0.85 | 91.30 | 2.90 | 0.85 | 0.00 | 0.34 | 0.00 | 0.00 | 3.75 |
| 3 | 1161 | 0.00 | 1.64 | 89.15 | 3.70 | 0.17 | 0.43 | 0.00 | 0.00 | 4.91 |
| 4 | 846 | 0.00 | 0.35 | 3.66 | 86.29 | 2.72 | 0.71 | 0.12 | 0.35 | 5.79 |
| 5 | 557 | 0.00 | 0.00 | 0.18 | 8.62 | 76.12 | 4.67 | 0.00 | 0.18 | 10.23 |
| 6 | 479 | 0.00 | 0.00 | 0.63 | 0.42 | 7.10 | 74.53 | 2.51 | 3.34 | 11.48 |
| 7 | 28 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 14.29 | 53.57 | 10.71 | 21.43 |

Table 3: Estimated thresholds and variances in the constrained probit

| | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| threshold | 0.0 | 6.27 | 6.55 | 7.42 | 8.46 | 12.53 | 12.95 |
| $\sigma^*$ | 1.00 | 0.43 | 0.02 | 0.28 | 0.343 | 1.41 | 0.21 |

Table 4: Estimation of the approximate factor model

| | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| $\delta^*$ | -1.65 | 5.65 | 6.52 | 7.01 | 7.92 | 10.70 | 12.81 |
| | (0.06) | (0.082) | (0.005) | (0.041) | (0.053) | (0.384) | (0.085) |
| $\gamma^*$ | 0.001 | 0.003 | 0.019 | 0.03 | 0.23 | 0.04 | 0.82 |
| | (1.0e-4) | (2.0e-4) | (2.11e-3) | (2.29e-3) | (1.62e-2) | (3.72e-3) | (0.121) |
| $\rho$ | 0.02 | | | | | | |
| | (0.011) | | | | | | |

Note: The standard errors are given in parentheses.

Table 5: Filtered factor values

| 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 |
|---|---|---|---|---|---|---|
| 0.967 | 0.653 | -0.602 | -1.192 | -0.625 | -0.730 | -1.607 |
| 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | |
| -0.852 | -0.03 | 0.493 | 0.774 | 1.095 | 1.607 | |

Table 6: SML estimation of one factor probit

| | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| $\delta^*$ | -1.59 | 5.66 | 6.77 | 7.32 | 8.25 | 10.95 | 12.71 |
| | (0.08) | (0.06) | (0.04) | (0.04) | (0.05) | (0.023) | (0.072) |
| $\gamma^*$ | 0.002 | 0.003 | 0.43 | 0.02 | 0.24 | 0.05 | 0.82 |
| | (1.0e-04) | (1.0e-04) | (0.01) | (2.0e-03) | (1.8e-02) | (4.0e-03) | (0.11) |
| $\rho$ | 0.01 | | | | | | |
| | (0.01) | | | | | | |

Table 7: Score dispersion

| Rating | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| marginal variance | 1.00 | 0.19 | 0.19 | 0.08 | 0.18 | 1.58 | 23.32 |
| Factor explained variance(in %) | 0.00 | .01 | 99.19 | 0.35 | 55.26 | 0.20 | 30.98 |

Table 8: Filtered factor values for factor probit

| 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 |
|---|---|---|---|---|---|---|
| 0.967 | 0.653 | -0.602 | -1.192 | -0.625 | -0.730 | -1.607 |
| 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | |
| -0.852 | -0.03 | 0.493 | 0.774 | 1.095 | 1.607 | |

Table 9: Term structure of correlations for migration BB $\to$ B, 2 firms

| horizon | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| correlation | -0.1094 | -0.0247 | -0.0091 | 0.028 |

| rating | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| correlation | -5.26e-06 | 0.000 | 0.028 | 6.00e-06 | 0.081 | 0.000 | 0.200 |

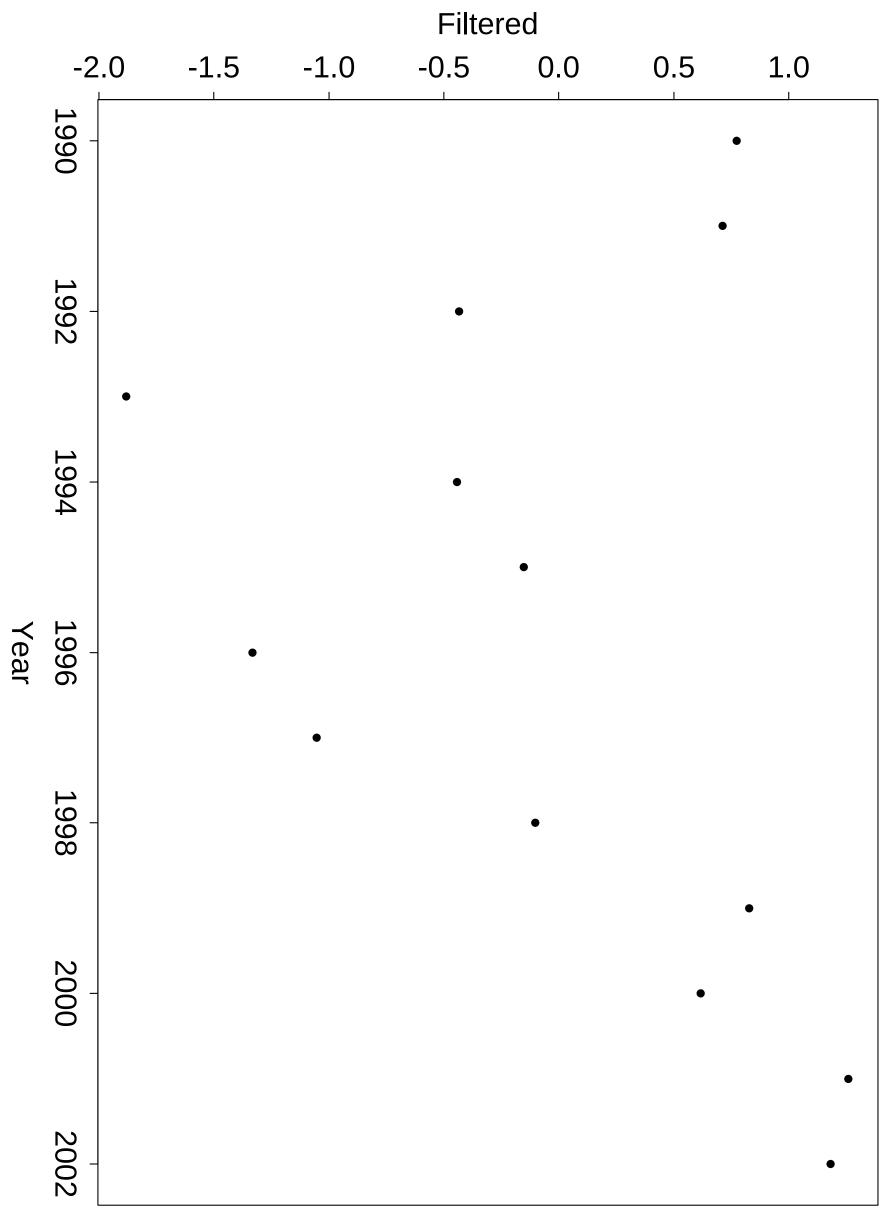
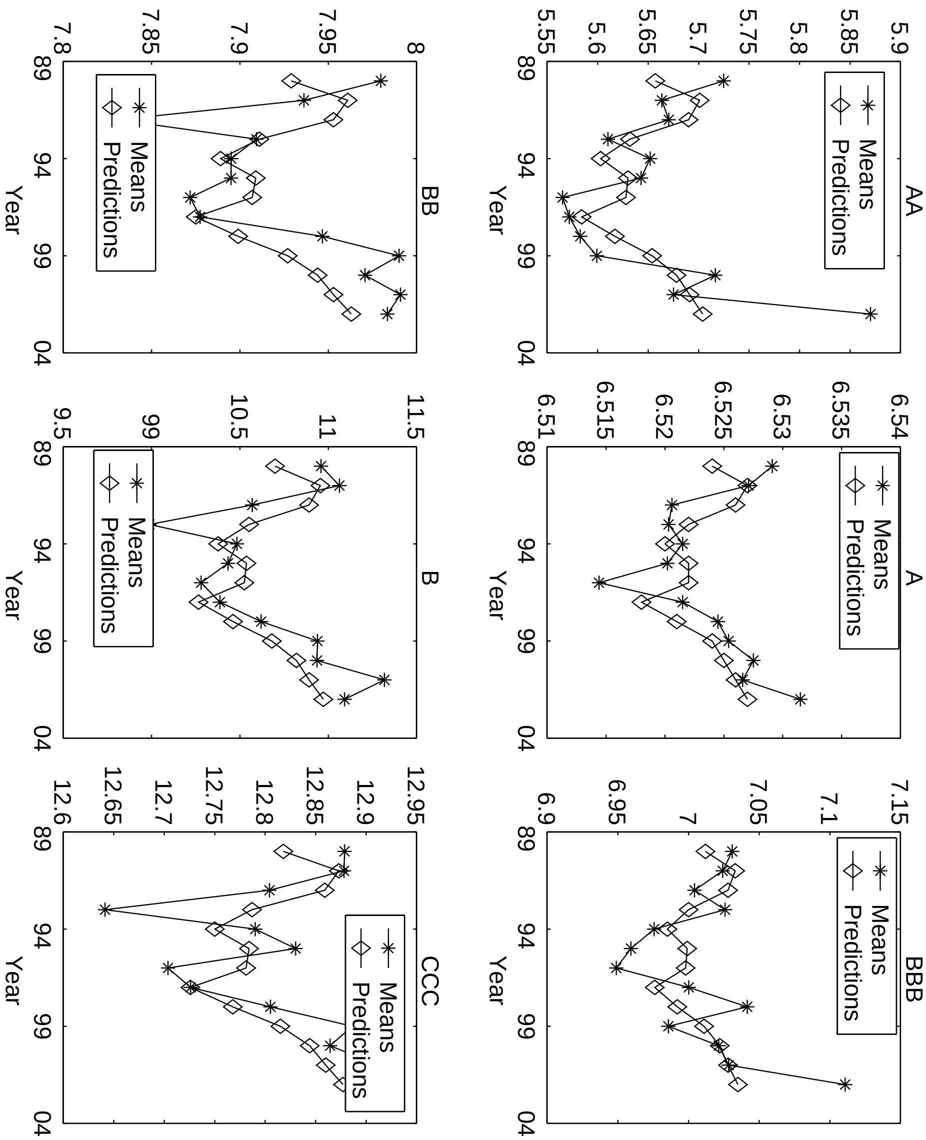

Filtered
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
Year
1990
1992
1994
1996
1998
2000
2002

**Figure 2: Means and fitted means**



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# Appendix 1: Additional Tables and Figures

Table a.1 Estimates of thresholds, $a^*_{1,t} = 0$

| Year | $a^*_2$ | $a^*_3$ | $a^*_4$ | $a^*_5$ | $a^*_6$ | $a^*_7$ |
|---|---|---|---|---|---|---|
| 1981 | 4.04 | 6.78 | 7.97 | 11.19 | 14.49 | 21.77 |
| 1982 | 5.40 | 5.71 | 6.39 | 7.40 | 12.53 | 13.11 |
| 1983 | 4.94 | 5.30 | 6.32 | 7.34 | 14.35 | 14.59 |
| 1984 | 4.57 | 5.06 | 6.02 | 7.03 | 13.57 | 13.84 |
| 1985 | 5.69 | 5.99 | 6.67 | 7.69 | 12.81 | 13.40 |
| 1986 | 6.11 | 6.86 | 8.29 | 9.31 | 15.65 | 16.24 |
| 1987 | 3.01 | 3.44 | 4.93 | 5.39 | 10.25 | 10.69 |
| 1988 | 6.35 | 6.58 | 7.49 | 8.51 | 12.44 | 12.82 |
| 1989 | 6.24 | 6.47 | 7.37 | 8.40 | 12.30 | 12.69 |
| 1990 | 6.35 | 6.58 | 7.49 | 8.51 | 12.43 | 12.82 |
| 1991 | 6.35 | 6.61 | 7.49 | 8.51 | 12.44 | 12.82 |
| 1992 | 6.34 | 6.80 | 7.46 | 8.47 | 12.48 | 13.05 |
| 1993 | 6.42 | 6.65 | 7.55 | 8.58 | 12.51 | 12.89 |
| 1994 | 5.66 | 6.10 | 6.85 | 7.86 | 12.62 | 13.16 |
| 1995 | 6.41 | 6.78 | 7.66 | 8.73 | 12.65 | 12.99 |
| 1996 | 5.97 | 6.30 | 7.08 | 8.09 | 12.35 | 12.99 |
| 1997 | 6.35 | 6.58 | 7.48 | 8.51 | 12.43 | 12.81 |
| 1998 | 6.34 | 6.57 | 7.47 | 8.50 | 12.43 | 12.81 |
| 1999 | 6.32 | 6.54 | 7.45 | 8.47 | 12.41 | 12.80 |
| 2000 | 6.27 | 6.49 | 7.40 | 8.43 | 12.36 | 12.75 |
| 2001 | 6.35 | 6.58 | 7.48 | 8.51 | 12.44 | 12.82 |
| 2002 | 6.35 | 6.58 | 7.49 | 8.51 | 12.43 | 12.82 |

Table a.2: Estimates of latent means

| Year | $m_1^*$ | $m_2^*$ | $m_3^*$ | $m_4^*$ | $m_5^*$ | $m_6^*$ | $m_7^*$ |
|---|---|---|---|---|---|---|---|
| 1981 | -1.36 | 2.92 | 5.50 | 7.37 | 10.44 | 11.29 | 14.53 |
| 1982 | -1.23 | 4.84 | 5.68 | 6.12 | 6.95 | 10.23 | 12.93 |
| 1983 | -0.88 | 4.31 | 5.25 | 5.81 | 6.89 | 11.01 | 14.42 |
| 1984 | - 0.47 | 3.93 | 4.97 | 5.53 | 6.45 | 10.17 | 13.82 |
| 1985 | - 1.47 | 5.22 | 5.72 | 6.40 | 7.23 | 10.50 | 12.97 |
| 1986 | -1.39 | 4.22 | 6.18 | 7.66 | 8.77 | 13.54 | 16.13 |
| 1987 | -1.67 | 2.44 | 3.32 | 4.27 | 5.16 | 7.89 | 10.49 |
| 1988 | -1.75 | 5.84 | 6.54 | 7.04 | 8.04 | 10.58 | 12.69 |
| 1989 | -1.60 | 5.54 | 6.44 | 6.93 | 7.76 | 10.38 | 12.67 |
| 1990 | - 1.75 | 5.75 | 6.55 | 7.07 | 8.04 | 10.92 | 12.75 |
| 1991 | - 1.75 | 5.68 | 6.55 | 7.07 | 8.04 | 11.01 | 12.75 |
| 1992 | - 1.31 | 5.71 | 6.55 | 7.14 | 7.86 | 10.56 | 12.85 |
| 1993 | - 1.77 | 5.69 | 6.60 | 7.14 | 8.03 | 10.03 | 12.61 |
| 1994 | - 1.37 | 5.06 | 5.83 | 6.46 | 7.30 | 10.25 | 12.94 |
| 1995 | - 1.73 | 5.78 | 6.72 | 7.19 | 8.15 | 10.62 | 12.89 |
| 1996 | - 1.48 | 5.67 | 6.26 | 6.64 | 7.52 | 10.01 | 12.60 |
| 1997 | -1.43 | 5.61 | 6.54 | 7.04 | 7.93 | 10.35 | 12.60 |
| 1998 | -1.73 | 5.60 | 6.53 | 7.07 | 8.03 | 10.58 | 12.68 |
| 1999 | - 1.73 | 5.61 | 6.46 | 6.94 | 7.96 | 10.82 | 12.69 |
| 2000 | - 1.74 | 5.74 | 6.55 | 7.06 | 8.04 | 10.89 | 12.74 |
| 2001 | -1.75 | 5.69 | 6.54 | 7.07 | 8.04 | 11.26 | 12.78 |
| 2002 | -1.75 | 5.90 | 6.55 | 7.15 | 8.04 | 11.04 | 12.78 |

Table a.3: Estimates of latent standard deviations ($\sigma^*_{1t} = 1$)

| Year | $\sigma^*_2$ | $\sigma^*_3$ | $\sigma^*_4$ | $\sigma^*_5$ | $\sigma^*_6$ | $\sigma^*_7$ |
|---|---|---|---|---|---|---|
| 1981 | 0.75 | 0.86 | 0.36 | 1.55 | 0.06 | 0.03 |
| 1982 | 0.43 | 0.02 | 0.22 | 0.37 | 1.65 | 0.28 |
| 1983 | 0.41 | 0.03 | 0.36 | 0.47 | 2.13 | 0.05 |
| 1984 | 0.41 | 0.04 | 0.44 | 0.38 | 1.90 | 0.01 |
| 1985 | 0.43 | 0.01 | 0.22 | 0.37 | 1.65 | 0.28 |
| 1986 | 1.42 | 0.04 | 0.59 | 0.35 | 2.27 | 0.12 |
| 1987 | 0.36 | 0.00 | 0.54 | 0.17 | 1.58 | 0.16 |
| 1988 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 1989 | 0.46 | 0.02 | 0.29 | 0.37 | 1.35 | 0.21 |
| 1990 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 1991 | 0.46 | 0.02 | 0.29 | 0.35 | 1.36 | 0.22 |
| 1992 | 0.45 | 0.01 | 0.24 | 0.34 | 1.65 | 0.29 |
| 1993 | 0.46 | 0.02 | 0.28 | 0.36 | 1.35 | 0.20 |
| 1994 | 0.42 | 0.02 | 0.20 | 0.31 | 1.57 | 0.25 |
| 1995 | 0.42 | 0.03 | 0.26 | 0.36 | 1.35 | 0.19 |
| 1996 | 0.18 | 0.01 | 0.22 | 0.36 | 1.43 | 0.24 |
| 1997 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 1998 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 1999 | 0.46 | 0.02 | 0.29 | 0.35 | 1.36 | 0.20 |
| 2000 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 2001 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |
| 2002 | 0.46 | 0.02 | 0.29 | 0.35 | 1.35 | 0.20 |

Table a.4: Estimated latent means when thresholds and variances are constant

| | Year | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|---|
| m | 1990 | -1.72 | 5.72 | 6.52 | 7.03 | 7.97 | 10.95 | 12.87 |
| e | 1991 | -1.73 | 5.66 | 6.52 | 7.02 | 7.93 | 11.06 | 12.87 |
| a | 1992 | -1.69 | 5.67 | 6.52 | 7.00 | 7.83 | 10.56 | 12.80 |
| n | 1993 | -1.42 | 5.61 | 6.52 | 7.02 | 7.90 | 9.98 | 12.64 |
| s | 1994 | - 1.72 | 5.65 | 6.52 | 6.97 | 7.89 | 10.48 | 12.79 |
| | 1995 | -1.51 | 5.64 | 6.52 | 6.95 | 7.89 | 10.43 | 12.83 |
| | 1996 | -1.62 | 5.56 | 6.51 | 6.94 | 7.87 | 10.28 | 12.70 |
| m | 1997 | -1.47 | 5.57 | 6.52 | 7.00 | 7.87 | 10.38 | 12.72 |
| e | 1998 | -1.73 | 5.58 | 6.52 | 7.04 | 7.94 | 10.62 | 12.80 |
| a | 1999 | -1.71 | 5.59 | 6.52 | 6.98 | 7.99 | 10.93 | 12.88 |
| n | 2000 | -1.69 | 5.71 | 6.52 | 7.02 | 7.97 | 10.93 | 12.86 |
| s | 2001 | -1.65 | 5.67 | 6.52 | 7.02 | 7.99 | 11.31 | 12.91 |
| | 2002 | -1.68 | 5.87 | 6.53 | 7.11 | 7.98 | 11.09 | 12.91 |

Table a.5: The prediction of transition matrix, $Q^{(1)}(H|f_T)$

| Horizon | | AAA | AA | A | BBB | BB | B | CCC | D |
|---|---|---|---|---|---|---|---|---|---|
| 1 year | AAA | 0.9505 | 0.0495 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | AA | 0.0000 | 0.9141 | 0.0847 | 0.0012 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | 0.0000 | 0.0334 | 0.8192 | 0.1472 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| | BBB | 0.0000 | 0.0001 | 0.1191 | 0.8432 | 0.0375 | 0.0000 | 0.0000 | 0.0000 |
| | BB | 0.0000 | 0.0000 | 0.0000 | 0.0487 | 0.7141 | 0.2372 | 0.0000 | 0.0000 |
| | B | 0.0000 | 0.0001 | 0.0007 | 0.0053 | 0.0177 | 0.7803 | 0.1441 | 0.0519 |
| | CCC | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.2014 | 0.4125 | 0.3860 |
| 2 year | AAA | 0.9034 | 0.0923 | 0.0042 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | AA | 0.0000 | 0.8381 | 0.1477 | 0.0141 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | 0.0000 | 0.0550 | 0.7060 | 0.2336 | 0.0054 | 0.0000 | 0.0000 | 0.0000 |
| | BBB | 0.0000 | 0.0038 | 0.1993 | 0.7295 | 0.0585 | 0.0088 | 0.0000 | 0.0000 |
| | BB | 0.0000 | 0.0000 | 0.0060 | 0.0753 | 0.5281 | 0.3463 | 0.0326 | 0.0117 |
| | B | 0.0000 | 0.0002 | 0.0017 | 0.0096 | 0.0268 | 0.6419 | 0.1719 | 0.1479 |
| | CCC | 0.0000 | 0.0000 | 0.0001 | 0.0011 | 0.0036 | 0.2418 | 0.2057 | 0.5476 |
| 5 year | AAA | 0.7758 | 0.1884 | 0.0307 | 0.0050 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| | AA | 0.0000 | 0.6598 | 0.2563 | 0.0802 | 0.0033 | 0.0005 | 0.0000 | 0.0000 |
| | A | 0.0000 | 0.1037 | 0.4968 | 0.3627 | 0.0278 | 0.0080 | 0.0007 | 0.0004 |
| | BBB | 0.0000 | 0.0283 | 0.3069 | 0.5355 | 0.0764 | 0.0422 | 0.0059 | 0.0048 |
| | BB | 0.0000 | 0.0015 | 0.0319 | 0.1085 | 0.2205 | 0.4029 | 0.0936 | 0.1411 |
| | B | 0.0000 | 0.0006 | 0.0060 | 0.0185 | 0.0325 | 0.4040 | 0.1279 | 0.4105 |
| | CCC | 0.0000 | 0.0001 | 0.0012 | 0.0048 | 0.0105 | 0.1805 | 0.0634 | 0.7395 |
| 10 year | AAA | 0.6019 | 0.2736 | 0.0886 | 0.0332 | 0.0019 | 0.0006 | 0.0001 | 0.0001 |
| | AA | 0.0000 | 0.4636 | 0.3207 | 0.1906 | 0.0159 | 0.0071 | 0.0010 | 0.0011 |
| | A | 0.0000 | 0.1298 | 0.3861 | 0.3885 | 0.0475 | 0.0330 | 0.0059 | 0.0093 |
| | BBB | 0.0000 | 0.0652 | 0.3266 | 0.4114 | 0.0673 | 0.0741 | 0.0162 | 0.0393 |
| | BB | 0.0000 | 0.0082 | 0.0589 | 0.1040 | 0.0721 | 0.2760 | 0.0789 | 0.4018 |
| | B | 0.0000 | 0.0019 | 0.0124 | 0.0241 | 0.0232 | 0.2023 | 0.0629 | 0.6732 |
| | CCC | 0.0000 | 0.0005 | 0.0036 | 0.0080 | 0.0093 | 0.0906 | 0.0285 | 0.8596 |

Table a.6 The key cells in $Q^{(2)}(H|f_T) - Q^{(1)}(H|f_T) \otimes Q^{(1)}(H|f_T)$ for horizon 1 year

| co-movement | | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|---|
| up 1 bucket | AAA | — | — | — | — | — | — | — |
| | AA | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | — | 0.0000 | 0.0298 | 0.0010 | 0.0231 | 0.0002 | 0.0274 |
| | BBB | — | 0.0000 | 0.0010 | 0.0002 | -0.0003 | 0.0000 | 0.0034 |
| | BB | — | 0.0000 | 0.0237 | 0.0012 | 0.0234 | 1e-04 | 0.0367 |
| | B | — | 0.0000 | 0.0002 | 0.0000 | -0.0004 | 0.0000 | 0.0006 |
| | CCC | — | 0.0000 | 0.0274 | 0.0034 | 0.0307 | 0.0006 | 0.1184 |
| unchanged | AAA | -0.0001 | -0.0001 | 0.0000 | 0.0000 | 0.0705 | 0.0000 | 0.0000 |
| | AA | -0.0001 | 0.0000 | 0.0001 | 0.0000 | 0.0608 | 0.0000 | 0.0000 |
| | A | 0.0000 | 0.0001 | 0.1300 | -0.0008 | 0.1627 | 0.0016 | 0.0733 |
| | BBB | 0.0000 | 0.0000 | -0.0008 | 0.0001 | -0.0616 | 0.0001 | -0.0005 |
| | BB | 0.0705 | 0.0680 | 0.1627 | 0.0616 | 0.2037 | 0.0599 | 0.1244 |
| | B | -1e-04 | -1e-04 | 0.0017 | 0.0000 | 0.0020 | 0.0000 | 0.0011 |
| | CCC | 0.0000 | 0.0000 | 0.0733 | -0.0005 | 0.1244 | 0.0011 | 0.1407 |
| down 1 bucket | AAA | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | AA | 0.0000 | 0.0000 | 0.0002 | 0.0000 | 0.0003 | 0.0000 | 0.0004 |
| | A | 0.0000 | 0.0003 | 0.1099 | 0.0011 | 0.0868 | 0.0018 | 0.0886 |
| | BBB | 0.0000 | 0.0000 | 0.0011 | 0.0000 | -0.0001 | 0.0000 | 0.0018 |
| | BB | -0.0019 | -0.003 | 0.0868 | -0.0001 | 0.0823 | -0.0034 | 0.1085 |
| | B | 0.0000 | 0.0000 | 0.0018 | 0.0000 | -0.0034 | 0.0001 | 0.0029 |
| | CCC | 0.0000 | 4e-04 | 0.0886 | 0.0018 | 0.1085 | 0.0029 | 0.1789 |

Table a.7: The key cells in $Q^{(2)}(H|f_T) - Q^{(1)}(H|f_T) \otimes Q^{(1)}(H|f_T)$ for 2-year horizon

| co-movement | | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|---|
| up 1 bucket | AAA | — | — | — | — | — | — | — |
| | AA | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | — | 0.0000 | 0.0445 | -0.0009 | 0.0292 | 0.0001 | 0.0260 |
| | BBB | — | 0.0000 | -0.0009 | 0.0023 | -0.0099 | 0.0005 | 0.0037 |
| | BB | — | 0.0000 | 0.0292 | -0.0099 | 0.0187 | -0.0011 | 0.0230 |
| | B | — | 0.0000 | 0.0001 | 0.0005 | -0.0011 | 0.0001 | 0.0009 |
| | CCC | — | 0.0000 | 0.0260 | 0.0037 | 0.0230 | 0.0009 | 0.1008 |
| unchanged | AAA | 0.0001 | 0.0000 | 0.0000 | -0.0012 | 0.0930 | -0.0006 | 0.0001 |
| | AA | 0.0000 | 0.0001 | -0.0014 | -0.0013 | 0.0856 | -0.0003 | -0.0003 |
| | A | 0.0000 | -0.0014 | 0.1678 | -0.0129 | 0.1915 | 0.0006 | 0.0513 |
| | BBB | -0.0012 | -0.0013 | -0.0129 | -0.0002 | 0.0655 | -0.0020 | -0.0039 |
| | BB | 0.0930 | 0.0856 | 0.1915 | 0.0655 | 0.2104 | 0.0678 | 0.0861 |
| | B | -0.0006 | -0.0003 | 0.0006 | -0.0020 | 0.0678 | 0.0018 | -0.0021 |
| | CCC | 0.0001 | -0.0003 | 0.0513 | -0.0039 | 0.0861 | -0.0021 | 0.0672 |
| | AAA | 0.0000 | 0.0000 | 0.0000 | 0.0003 | -0.0037 | 0.0000 | -0.0001 |
| down 1 bucket | AA | 0.0000 | 0.0009 | -0.0070 | 0.0007 | -0.0107 | 0.0008 | -0.0024 |
| | A | 0.0000 | -0.0070 | 0.1425 | -0.0005 | 0.0907 | -0.0053 | 0.0821 |
| | BBB | 0.0003 | 0.0007 | -0.0005 | 0.0004 | -0.0021 | 0.0009 | 0.0017 |
| | BB | -0.0037 | -0.0107 | 0.0907 | -0.0021 | 0.0672 | -0.0131 | 0.0817 |
| | B | 0.0000 | 0.0008 | -0.0053 | 0.0009 | -0.0131 | 0.0028 | -0.0050 |
| | CCC | -0.0001 | -0.0024 | 0.0821 | 0.0017 | 0.0817 | -0.0050 | 0.1685 |

Table a.8: The key cells in $Q^{(2)}(H|f_T) - Q^{(1)}(H|f_T) \otimes Q^{(1)}(H|f_T)$ for 5-year horizon

| co-movement | | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|---|
| up 1 bucket | AAA | — | — | — | — | — | — | — |
| | AA | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | — | 0.0000 | 0.0643 | -0.0080 | 0.0287 | 0.0002 | 0.0174 |
| | BBB | — | 0.0000 | -0.0080 | 0.0191 | -0.0172 | 0.0034 | 0.0030 |
| | BB | — | 0.0000 | 0.0287 | -0.0172 | 0.0124 | -0.0006 | 0.0124 |
| | B | — | 0.0000 | 0.0002 | 0.0034 | -0.0006 | 0.0005 | 0.0018 |
| | CCC | — | 0.0000 | 0.0174 | 0.0030 | 0.0124 | 0.0018 | 0.0454 |
| unchanged | AAA | 0.0000 | 0.0000 | 0.0000 | -0.0045 | 0.0623 | -0.0011 | 0.0000 |
| | AA | 0.0000 | 0.0036 | -0.0070 | -0.0045 | 0.0508 | 0.0013 | -0.0002 |
| | A | 0.0000 | -0.0070 | 0.1413 | -0.0385 | 0.1110 | 0.0015 | 0.0113 |
| | BBB | -0.0045 | -0.0045 | -0.0385 | 0.0094 | 0.0249 | -0.0052 | -0.0035 |
| | BB | 0.0623 | 0.0508 | 0.1110 | 0.0249 | 0.0813 | 0.0363 | 0.0177 |
| | B | -0.0011 | 0.0013 | 0.0015 | -0.0052 | 0.0363 | 0.0065 | 0.0006 |
| | CCC | 0.0000 | -0.0002 | 0.0113 | -0.0035 | 0.0177 | 0.0006 | 0.0079 |
| down 1 bucket | AAA | 0.0000 | -0.0002 | -0.0004 | 0.0021 | 0.0005 | -0.0002 | -0.0001 |
| | AA | -0.0002 | 0.0113 | -0.0258 | 0.0053 | -0.0090 | 0.0033 | -0.0039 |
| | A | -0.0004 | -0.0258 | 0.1216 | -0.0007 | 0.0531 | -0.0087 | 0.0385 |
| | BBB | 0.0021 | 0.0053 | -0.0007 | 0.0024 | 0.0023 | 0.0028 | 0.0076 |
| | BB | 0.0005 | -0.0090 | 0.0531 | 0.0023 | 0.0404 | -0.0062 | 0.0360 |
| | B | -0.0002 | 0.0033 | -0.0087 | 0.0028 | -0.0062 | 0.0046 | -0.0048 |
| | CCC | -0.0001 | -0.0039 | 0.0385 | 0.0076 | 0.0360 | -0.0048 | 0.0893 |

Table a.9: The key cells in $Q^{(2)}(H|f_T) - Q^{(1)}(H|f_T) \otimes Q^{(1)}(H|f_T)$ for 10-year horizon

| co-movement | | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|---|
| up 1 bucket | AAA | — | — | — | — | — | — | — |
| | AA | — | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | A | — | 0.0000 | 0.0531 | -0.0097 | 0.0145 | 0.0007 | 0.0070 |
| | BBB | — | 0.0000 | -0.0097 | 0.0354 | -0.0166 | 0.0041 | 0.0002 |
| | BB | — | 0.0000 | 0.0145 | -0.0166 | 0.0072 | 0.0000 | 0.0052 |
| | B | — | 0.0000 | 0.0007 | 0.0041 | 0.0000 | 0.0005 | 0.0009 |
| | CCC | — | 0.0000 | 0.0070 | 0.0002 | 0.0052 | 0.0009 | 0.0122 |
| unchanged | AAA | 0.0000 | 0.0000 | -0.0007 | -0.0055 | 0.0175 | 0.0005 | 0.0000 |
| | AAA | 0.0000 | 0.0125 | -0.0100 | -0.0083 | 0.0123 | 0.0030 | 0.0000 |
| | AA | -0.0007 | -0.0100 | 0.0763 | -0.0422 | 0.0293 | 0.0013 | 0.0020 |
| | A | -0.0055 | -0.0083 | -0.0422 | 0.0232 | 0.0033 | -0.0030 | -0.0014 |
| | BBB | 0.0175 | 0.0123 | 0.0293 | 0.0033 | 0.0114 | 0.0076 | 0.0024 |
| | BB | 0.0005 | 0.0030 | 0.0013 | -0.0030 | 0.0076 | 0.0047 | 0.0006 |
| | B | 0.0000 | 0.0000 | 0.0020 | -0.0014 | 0.0024 | 0.0006 | 0.0017 |
| down 1 bucket | CCC | 0.0004 | -0.0013 | -0.0025 | 0.0036 | 0.0086 | -0.0001 | -0.0002 |
| | AA | -0.0013 | 0.0308 | -0.0351 | 0.0083 | 0.0068 | 0.0026 | -0.0018 |
| | A | -0.0025 | -0.0351 | 0.0677 | 0.0021 | 0.0220 | -0.0039 | 0.0056 |
| | BBB | 0.0036 | 0.0083 | 0.0021 | 0.0026 | 0.0051 | 0.0016 | 0.0119 |
| | BB | 0.0086 | 0.0068 | 0.0220 | 0.0051 | 0.0260 | 0.0011 | 0.0289 |
| | B | -0.0001 | 0.0026 | -0.0039 | 0.0016 | 0.0011 | 0.0014 | -0.0017 |
| | CCC | -0.0002 | -0.0018 | 0.0056 | 0.0119 | 0.0289 | -0.0017 | 0.0282 |