# Possibility and permissibility

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**Abstract**

We generalize permissibility (Brandenburger, 1992) to allow for any suitably defined model of preference and definition of possibility. We also prove that the generalized solution concept characterizes rationality, caution, and common "belief" of rationality and caution.

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## 1 Introduction

The expected utility model of Savage (1954) is the standard theory of decision under uncertainty. An important property underlying the model is, in Machina and Schmeidler's (1992) terminology, probabilistic sophistication. A decision maker is probabilistically sophisticated if his preference reflects probabilistic beliefs, in the sense that events are distinguished only by (subjective) probabilities assigned to them.

Savage's axiomatization of expected utility includes a preference-based definition of possibility. The definition, when adapted to any finite state space, says that a state is nonnull if the decision maker is ever concerned about his consequence at that state. Nonnullity fits into the traditional recipes of defining possible states in terms of believed events or possible events.

However, the concept of nonnullity is arguably too loose. For non-probabilistically sophisticated preferences, there are motivations (especially in game theory) to develop stronger definitions, so that a nonnull state is not necessarily classified as possible. Inspired by Morris (1997), Ryan (2002), and Lo (2005a), who formulate definitions targeting specific models of preference, Lo (2005b) proposes a general recipe, which can be easily used to formulate desired preference-based definitions of possibility. His recipe includes the traditional recipes as special cases.

In this paper, we adopt a more comprehensive version of Lo's recipe, and derive how possible states in the "grand world" are related to possible states in every "small world." Roughly speaking, every possible state in every small world contains a possible state in the grand world; but unless the recipe collapses to the traditional ones, a small-world state may

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not be possible, even though the small-world state contains a possible grand-world state. These results are relevant for defining—with foundations—solution concepts in games with non-probabilistically sophisticated players. The focus of this paper is permissibility, which was formulated by Brandenburger (1992) in terms of lexicographic expected utility.[1] We generalize permissibility to allow for any suitably defined model of preference and definition of possibility. We also use the results to prove that the generalized solution concept characterizes rationality, caution, and common "belief" of rationality and caution.

The following (notational) conventions will be adopted throughout the paper. The set of consequences is always $\mathbb{R}$, but various state spaces (which are all assumed to be finite) will arise. For any set $Z$ of states, use $\mathcal{F}(Z)$ to denote the set of acts (i.e., functions) from $Z$ to $\mathbb{R}$. As is customary, for any $c \in \mathbb{R}$, $c$ also denotes the constant act that yields the consequence $c$ in every state $z \in Z$; for any $c, d \in \mathbb{R}$, and any $\zeta \subseteq Z$, $c\zeta d$ denotes the binary act that yields $c$ if the event $\zeta$ happens, and yields $d$ if $Z \setminus \zeta$ happens.[2] For any preference (i.e., complete and transitive) relation $\succeq_Z$ on $\mathcal{F}(Z)$, use $\succ_Z$ to denote the asymmetric part of $\succeq_Z$. Any preference relation considered in this paper is also assumed to be *weakly monotonic* (i.e., for all $f, f' \in \mathcal{F}(Z)$, if $f(z) \geq f'(z)$ for all $z \in Z$, then $f \succeq_Z f'$) and *non-trivial* (i.e., there exist $f, f' \in \mathcal{F}(Z)$ such that $f \succ_Z f'$).

# 2 Defining possibility in grand and small worlds

Fix a set $T$ of states; call it the *grand world*. Call any partition $S$ of $T$ a *small world*. (For notational simplicity, $T$ also denotes the finest small world, namely, the small world in which every partitional element is a singleton). It is convenient to think of a grand-world state as a completely detailed description of the world, leaving no relevant aspect undescribed. While a small-world state (which is virtually a grand-world event) is in general not completely detailed, it may still be—in the context of a certain decision problem—sufficiently detailed, leaving no payoff-relevant aspect undescribed. A simple example (which will be carefully considered in Section 4) is as follows. Alice is playing a strategic game with Bob. From the perspective of Alice, $T$ is the set of types, and $S$ is the set of strategies, of Bob. In this context, a grand-world state completely specifies Bob's strategy as well as state of mind; it is much more detailed than a small-world state, which is just a specification of Bob's strategy.

For any small world $S$, and any grand-world state $t \in T$, let $\mathbf{s}(t) \in S$ be the small-world state such that $t \in \mathbf{s}(t)$; in other words, $\mathbf{s}(t)$ is the partitional element containing $t$. Given any preference relation $\succeq$ on the set $\mathcal{F}(T)$ of acts over $T$, let $\succeq_S$ be the preference relation on the set $\mathcal{F}(S)$ of acts over $S$, which is derived from $\succeq$ as follows: For all $f, f' \in \mathcal{F}(S)$,

$$f \succeq_S f' \quad \text{if} \quad \tilde{f} \succeq \tilde{f}', \tag{1}$$

where $\tilde{f}(t) = f(\mathbf{s}(t))$ and $\tilde{f}'(t) = f'(\mathbf{s}(t))$ for all $t \in T$. In essence, $\succeq_S$ is virtually the restriction of $\succeq$ to acts which are measurable with respect to $S$. (So, when $S$ is the finest small world, it is understandable to identify $\succeq_S$ with $\succeq$.) We emphasize that $\succeq_S$ may not be an expected utility preference relation.

[1] See also Börgers (1994) and Dekel and Fudenberg (1990).

[2] In this paper, we use the symbol $\subseteq$ for subset, $\subset$ for proper subset, and $Z \setminus \zeta$ for the complement of $\zeta$.

For any small world $S$, and any $\sigma \subseteq S$, say that $\sigma$ is a $\succeq_S$-*nonnull event* if there exist acts $f, f' \in \mathcal{F}(S)$ such that $f(s) = f'(s)$ for all $s \in S \setminus \sigma$, and $f \succ_S f'$; otherwise $\sigma$ is a $\succeq_S$-*null event*. Intuitively, $\sigma$ is $\succeq_S$-nonnull if the decision maker (with preference represented by $\succeq_S$) is ever concerned about what he will receive at states lying inside $\sigma$. For any $s \in S$, say that $s$ is a $\succeq_S$-*nonnull state* if $\{s\}$ is a $\succeq_S$-nonnull event; similarly, $s$ is a $\succeq_S$-*null state* if $\{s\}$ is a $\succeq_S$-null event. Given the finite nature of $S$, an event is $\succeq_S$-nonnull if and only if it contains a $\succeq_S$-nonnull state. Weak monotonicity and non-triviality imply that $S$ is $\succeq_S$-nonnull, and so there must be a $\succeq_S$-nonnull state.

*A definition of likely events* $\mathcal{L}$ specifies, for every small world $S$, a collection $\mathcal{L}(\succeq_S)$ of events satisfying

**L1** For all $\sigma' \subseteq \sigma \subseteq S$, if $\sigma' \in \mathcal{L}(\succeq_S)$, then $\sigma \in \mathcal{L}(\succeq_S)$.

**L2** For all $\sigma \subseteq S$, if $\sigma \in \mathcal{L}(\succeq_S)$, then $\sigma$ is $\succeq_S$-nonnull.

**L3** For all $\sigma \subseteq S$ and all $s \in S$, if $\sigma \in \mathcal{L}(\succeq_S)$ and $s$ is $\succeq_S$-null, then $\sigma \setminus \{s\} \in \mathcal{L}(\succeq_S)$.

**L4** $S \in \mathcal{L}(\succeq_S)$.

**L5** For all $\sigma \subseteq S$, $\sigma \in \mathcal{L}(\succeq_S)$ if and only if $\mathbf{s}^{-1}(\sigma) \equiv \{t \in T | \mathbf{s}(t) \in \sigma\} \in \mathcal{L}(\succeq)$.

Properties L1 through L5 can be described as follows. L1: Any superset of a likely event is a likely event; L2: An event is likely only if it is nonnull; L3: A likely event with any null state removed is still a likely event; L4: The world is a likely event; L5: An event is likely if and only if the same event in the grand-world is likely.[3]

Given $\mathcal{L}(\succeq_S)$, a possible state is defined as follows.

**Definition 1.** A state $s \in S$ is $\mathcal{L}(\succeq_S)$-*possible* if there exists $\sigma \subseteq S$ such that $\sigma \notin \mathcal{L}(\succeq_S)$ and $\{s\} \cup \sigma \in \mathcal{L}(\succeq_S)$.

To elaborate, a state $s$ is $\mathcal{L}(\succeq_S)$-possible if there exists an event $\sigma$ such that $s$ has the following impact on $\sigma$: $\sigma$ is not a likely event, but $\sigma$ with $\{s\}$ attached becomes a likely event. Let

$$\min \mathcal{L}(\succeq_S) = \{\sigma \in \mathcal{L}(\succeq_S) | \text{For all } \sigma' \subset \sigma, \sigma' \notin \mathcal{L}(\succeq_S)\}$$

be the collection of *minimal likely events*. Properties L1–L4 imply that $\mathcal{L}(\succeq_S)$-possible states in every world are well behaved in the following sense (cf. Lo, 2005b, Propositions 1 and 2).

**Proposition 1.** *A state is $\mathcal{L}(\succeq_S)$-possible only if it is $\succeq_S$-nonnull. A state is $\mathcal{L}(\succeq_S)$-possible if and only if it is contained in $\cup_{\sigma \in \min \mathcal{L}(\succeq_S)} \sigma$. There exists at least one $\mathcal{L}(\succeq_S)$-possible state.*

In addition, L1–L4 also imply the following relationship between likely events and possible states: Any event containing all $\mathcal{L}(\succeq_S)$-possible states is a likely event, and any likely event contains at least one $\mathcal{L}(\succeq_S)$-possible state.

The remainder of this section is an example illustrating the above concepts. Let $a\colon 2^T \setminus \{\emptyset\} \to [0,1]$ be a function satisfying $\sum_{\tau \in 2^T \setminus \{\emptyset\}} a(\tau) = 1$; Shafer (1976) calls this a *basic probability assignment* on $T$. For each $\tau \in 2^T \setminus \{\emptyset\}$, the *basic probability number* $a(\tau)$ can

[3] As Lo (2005b) focuses on only one world, he is not explicit about L5.

be interpreted as the weight of evidence that the event $\tau$ has happened. Let $u\colon \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing von Neuman Morgenstern (vNM) index. Let

$$U(\tilde{f}) = \sum_{\tau \in 2^T \setminus \{\emptyset\}} a(\tau) \min_{t \in \tau} u(\tilde{f}(t)) \tag{2}$$

be the "expected minimum utility" of a typical grand-world act $\tilde{f} \in \mathcal{F}(T)$. Clearly, expected utility is a special case of Eq. (2), in the sense that $a(\tau) > 0$ only if $\tau$ is a singleton. Eq. (2) is in turn an important special case of two influential models: Choquet expected utility and maxmin expected utility (cf. Gilboa and Schmeidler, 1994). For any small world $S$, and any small-world act $f \in \mathcal{F}(S)$, if we let $\tilde{f}(t) = f(\mathbf{s}(t))$ for all $t \in T$, then for each $\tau \in 2^T \setminus \{\emptyset\}$,

$$\min_{t \in \tau} u(\tilde{f}(t)) = \min_{s \in \sigma} u(f(s)), \tag{3}$$

where $\sigma \in 2^S \setminus \{\emptyset\}$ is the smallest event such that $\tau \subseteq \mathbf{s}^{-1}(\sigma)$. Suppose the preference relation $\succeq$ on $\mathcal{F}(T)$ is represented by Eq. (2). Then it follows from Eq. (3) that the induced preference relation $\succeq_S$ as defined in Eq. (1) can be represented by, for every $f \in \mathcal{F}(S)$,

$$U_S(f) = \sum_{\sigma \in 2^S \setminus \{\emptyset\}} b(\sigma) \min_{s \in \sigma} u(f(s)), \tag{4}$$

where

$$b(\sigma) = \sum_{\{\tau \subseteq \mathbf{s}^{-1}(\sigma) | \text{For all } \sigma' \subset \sigma, \tau \not\subseteq \mathbf{s}^{-1}(\sigma')\}} a(\tau). \tag{5}$$

The function $b\colon 2^S \setminus \{\emptyset\} \to [0,1]$ in Eq. (5) is a basic probability assignment on $S$. (Clearly, if $S$ is the finest small world, then $b = a$.) Call

$$\text{supp } b = \{\sigma \in 2^S \setminus \{\emptyset\} | b(\sigma) > 0\} \tag{6}$$

the *support of* $b$. It is immediate from Eq. (4) that

$$\{s \in S | s \text{ is } \succeq_S\text{-nonnull}\} = \bigcup_{\sigma \in \text{supp } b} \sigma. \tag{7}$$

In words, a state is $\succeq_S$-nonnull if and only if it is covered by the support of $b$.

For every small world $S$, define

$$\mathcal{L}(\succeq_S) = \{\sigma \subseteq S | c\sigma d \succ_S d\sigma c' \text{ for all } c', c, d \in \mathbb{R} \text{ such that } c' > c > d\}. \tag{8}$$

Eq. (8) says that $\sigma \in \mathcal{L}(\succeq_S)$ if the event $\sigma$ is *infinitely more likely than* its complement $S \setminus \sigma$, in the sense that the decision maker strictly prefers to bet on $\sigma$ rather than on $S \setminus \sigma$, no matter how much better is the consequence for winning the $S \setminus \sigma$ bet than that for winning the $\sigma$ bet (cf. Lo, 1999). It is easy to check that this $\mathcal{L}$ satisfies L1–L5 (for any weakly monotonic and non-trivial preference), and therefore it is indeed a definition of likely events. Given that $\succeq_S$ is represented by Eq. (4), we have

$$\{s \in S | s \text{ is } \mathcal{L}(\succeq_S)\text{-possible}\} = \bigcup_{\sigma \in \min \text{supp } b} \sigma, \tag{9}$$

where

$$\min \operatorname{supp} b = \{\sigma \in 2^S \setminus \{\emptyset\} | b(\sigma) > 0, \text{ and for all } \sigma' \subset \sigma, b(\sigma') = 0\} \tag{10}$$

is called the *minimal support of* $b$. In words, a state is $\mathcal{L}(\succeq_S)$-possible if and only if it is covered by the minimal support of $b$.[4] Eqs. (7) and (9) imply that every $\mathcal{L}(\succeq_S)$-possible state is $\succeq_S$-nonnull (but the converse does not hold); Eqs. (9) and (10) imply that there exists at least one $\mathcal{L}(\succeq_S)$-possible state.

# 3 Relating possibility in grand and small worlds

In Section 2, we imposed L5, which determines likely events in a small world from likely events in the grand world. We now *derive* the relationship of possible states in the grand world and possible states in a small world. Proposition 2 below says that every possible state in every small world contains at least one possible state in the grand world.[5]

**Proposition 2.** *For every small world $S$, and every small-world state $s \in S$, if $s$ is $\mathcal{L}(\succeq_S)$-possible, then there exists a grand-world state $t \in \mathbf{s}^{-1}(s) \equiv \{t' \in T | \mathbf{s}(t') = s\}$ such that $t$ is $\mathcal{L}(\succeq)$-possible.*

Before asking the natural question of whether the converse of Proposition 2 holds, let us present the traditional recipes of defining possible states as special cases of the one in Section 2. Say that a definition of likely events $\mathcal{L}$ is also *a definition of believed events* if for every small world $S$, $\mathcal{L}(\succeq_S)$ also satisfies

**B1** For all $\sigma, \sigma' \subseteq S$, if $\sigma \in \mathcal{L}(\succeq_S)$ and $\sigma' \in \mathcal{L}(\succeq_S)$, then $\sigma \cap \sigma' \in \mathcal{L}(\succeq_S)$.

Obviously, B1 holds if and only if $\min \mathcal{L}(\succeq_S)$ is a singleton. By Proposition 1, $\min \mathcal{L}(\succeq_S)$ is a singleton if and only if

$$\min \mathcal{L}(\succeq_S) = \{\{s | s \text{ is } \mathcal{L}(\succeq_S)\text{-possible}\}\}. \tag{11}$$

In other words, $\mathcal{L}$ is a definition of believed events if and only if for every small world $S$, any likely event contains all $\mathcal{L}(\succeq_S)$-possible states.

Along the same line, say that a definition of likely events $\mathcal{L}$ is also *a definition of possible events* if for every small world $S$, $\mathcal{L}(\succeq_S)$ also satisfies

**P1** For all $\sigma \subseteq S$, if $\sigma \in \mathcal{L}(\succeq_S)$, then there exists $s \in \sigma$ such that $\{s\} \in \mathcal{L}(\succeq_S)$.

[4] According to Lo (2006), for any $\succeq_S$ representable by Eq. (4), the set of $\mathcal{L}(\succeq_S)$-possible states derived from Eq. (8) is the same as that derived from

$$\mathcal{L}(\succeq_S) = \{\sigma \subseteq S | c\sigma d \succ_S d \text{ for all } c, d \in \mathbb{R} \text{ such that } c > d\}.$$

In terms of this $\mathcal{L}(\succeq_S)$, it is straightforward that $\sigma \in \mathcal{L}(\succeq_S)$ if and only if there exists $\sigma' \subseteq \sigma$ such that $\sigma' \in \operatorname{supp} b$; so $\sigma \in \min \mathcal{L}(\succeq_S)$ if and only if $\sigma \in \min \operatorname{supp} b$. Recall that the set of $\mathcal{L}(\succeq_S)$-possible states is the union of all minimal likely events; Eq. (9) follows.

[5] Proof of Propositions 2–5 can be found in the Appendix.

Obviously, P1 holds if and only if every event in $\min \mathcal{L}(\succeq_S)$ is a singleton; or equivalently, by Proposition 1,

$$\min \mathcal{L}(\succeq_S) = \{\{s\} | s \text{ is } \mathcal{L}(\succeq_S)\text{-possible}\}. \tag{12}$$

So $\mathcal{L}$ is a definition of possible events if and only if for every small world $S$, any event containing at least one $\mathcal{L}(\succeq_S)$-possible state is a likely event.

It is well known that nonnull states can be derived from either a definition of believed events or a definition of possible events. For example,

$$\mathcal{L}(\succeq_S) = \{\sigma | S \setminus \sigma \text{ is } \succeq_S\text{-null}\} \tag{13}$$

is a collection of events satisfying B1, whereas

$$\mathcal{L}(\succeq_S) = \{\sigma | \sigma \text{ is } \succeq_S\text{-nonnull}\} \tag{14}$$

is a collection of events satisfying P1. Given $\mathcal{L}(\succeq_S)$ as defined in either Eq. (13) or Eq. (14), a state is $\mathcal{L}(\succeq_S)$-possible if and only if it is $\succeq_S$-nonnull.[6]

It is difficult to come up with a preference-based collection $\mathcal{L}(\succeq_S)$ satisfying B1/P1, such that not every $\succeq_S$-nonnull state is $\mathcal{L}(\succeq_S)$-possible; without the constraint of B1/P1, such a $\mathcal{L}(\succeq_S)$ can be easily formulated (cf. Lo, 2005b, 2006). Proposition 3 below reveals that the converse of Proposition 2 is equivalent to adding B1/P1 to $\mathcal{L}$.

**Proposition 3.** *The following two statements are equivalent:*

*(i) For every small world $S$, and every small-world state $s \in S$, if there exists a grand-world state $t \in \mathbf{s}^{-1}(s)$ such that $t$ is $\mathcal{L}(\succeq)$-possible, then $s$ is $\mathcal{L}(\succeq_S)$-possible.*

*(ii) The definition of likely events $\mathcal{L}$ is also a definition of believed events, or a definition of possible events.*

Let us use a story to heuristically illustrate both Propositions 2 and 3. As in Ellsberg (1961), suppose the decision maker is informed that an urn contains 90 balls, identical except in color; 30 of the balls are red, and each of the remaining balls is either green or yellow, but the relative proportions are unknown. One ball has been drawn from the urn, and the decision maker is interested in whether the color of that ball is red, green, or yellow. Let $T = \{t^r, t^g, t^y\}$ be the grand world (where $t^r$, $t^g$, and $t^y$ are the states in which the color of the ball is red, green, and yellow, respectively). Since $t^r$ has probability 1/3, it should be possible. However, $t^r$ should not be the only possible state. After all, $\{t^g, t^y\}$ is twice as probable as $\{t^r\}$. Since $t^g$ and $t^y$ are (informationally) symmetric, if any one of them is possible, then the other one should be as well. This intuition suggests that the decision maker may regard every state in $T$ as possible. Now consider the small world $S = \{s^{rg}, s^y\}$, where $s^{rg} = \{t^r, t^g\}$ and $s^y = \{t^y\}$. Intuitively, since there could be no yellow ball in the urn, $s^{rg}$ is "infinitely more likely than" $s^y$. So the decision maker may regard $s^{rg}$ as the only possible state (while $s^y$ is still nonnull) in $S$.

[6] More generally, for any definition of believed (possible, respectively) events, there is a definition of possible (believed, respectively) events generating the same set of possible states in every world. In this sense, the two concepts are equivalent.

The expected minimum utility example of Section 2 can be consistent with the above story. Let the basic probability assignment associated with Eq. (2) be

$$a(\tau) = \begin{cases} \frac{1}{3} & \text{if } \tau = \{t^r\} \\ \frac{2}{3} & \text{if } \tau = \{t^g, t^y\} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

By Eq. (5),

$$b(\sigma) = \begin{cases} \frac{1}{3} & \text{if } \sigma = \{s^{rg}\} \\ \frac{2}{3} & \text{if } \sigma = S \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Eqs. (9), (10), (15), and (16) imply that the set of $\mathcal{L}(\succeq)$-possible states is $T$, and the set of $\mathcal{L}(\succeq_S)$-possible states is $\{s^{rg}\}$. The $\mathcal{L}(\succeq_S)$-possible state $s^{rg}$ contains the $\mathcal{L}(\succeq)$-possible states $t^r$ and $t^g$. The state $s^y$ is not $\mathcal{L}(\succeq_S)$-possible, even though it contains the $\mathcal{L}(\succeq)$-possible state $t^y$. This is because, for non-probabilistically sophisticated preferences in general, likely events as defined in Eq. (8) are neither believed events nor possible events.

# 4 Generalizing permissibility in games

We now apply the decision theory of Sections 2 and 3 to game theory. Suppose that there are two players, 1 and 2.[7] They are playing a strategic game $(S_1, S_2, g_1, g_2)$, where $S_i$ is player $i$'s finite set of strategies, and $g_i\colon S_i \times S_j \to \mathbb{R}$ specifies $i$'s consequence for each strategy profile.[8] Since player $i$ may not know the strategy choice of player $j$, we suppose that player $i$ views the set $S_j$ as a small world. Fix a definition of likely events $\mathcal{L}$. So, if $i$'s preference relation on $\mathcal{F}(S_j)$ is $\succeq_i$, we use $\mathcal{L}(\succeq_i) \subseteq 2^{S_j}$ to denote the collection of likely events (satisfying L1–L4 of Section 2, with $\succeq_i$ in place of $\succeq_S$). Let $\mathcal{M}$ be *a model of preference*; for our purpose, $\mathcal{M}$ specifies, for every $i$, a set $\mathcal{M}(S_j)$ of preference relations on $\mathcal{F}(S_j)$ with the property: For every nonempty $\sigma_j \subseteq S_j$, there exists $\succeq_i \in \mathcal{M}(S_j)$ such that $s_j$ is $\succeq_i$-nonnull for all $s_j \in S_j$, but $s_j$ is $\mathcal{L}(\succeq_i)$-possible only if $s_j \in \sigma_j$.[9] Every strategy $s_i \in S_i$ can be interpreted as an element of $\mathcal{F}(S_j)$, in the sense that for each $s_j \in S_j$, $s_i$ delivers the consequence $g_i(s_i, s_j)$; under this interpretation, condition (i) of Definition 2 is meaningful.

**Definition 2.** A set $P_1 \times P_2 \subseteq S_1 \times S_2$ of strategy profiles is a $\langle \mathcal{L}, \mathcal{M} \rangle$*-permissible set* if for each $s_i \in P_i$, there exists $\succeq_i \in \mathcal{M}(S_j)$ such that the following three conditions are satisfied:

(i) $s_i \succeq_i s_i'$ for all $s_i' \in S_i$.

(ii) $s_j$ is $\succeq_i$-nonnull for all $s_j \in S_j$.

(iii) $s_j$ is $\mathcal{L}(\succeq_i)$-possible only if $s_j \in P_j$.

[7]Our analysis can be extended in a straightforward manner to games with more than two players.

[8]Unless emphasis is desired, it is understood that $i$ and $j$ vary over $\{1, 2\}$ and $i \neq j$.

[9]This property immediately implies the existence of $\succeq_i$ satisfying conditions (ii) and (iii) of Definition 2 below.

The three conditions in Definition 2 say that (i) $\succeq_i$ justifies $s_i$; (ii) $\succeq_i$ does not completely rule out any $s_j$; (iii) for every $s_j$ that is $\mathcal{L}(\succeq_i)$-possible, there exists $\succeq_j \in \mathcal{M}(S_i)$ such that $\succeq_j$ justifies $s_j$, $\succeq_j$ does not completely rule out any $s_i'$, and so on. By definition, $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible sets are closed under union; so the largest $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set exists. Let $P_i^0 = S_i$, and recursively define, for each positive integer $n$,

$$\begin{aligned} P_i^n = \{ & s_i \in S_i | \exists \succeq_i \in \mathcal{M}(S_j) \text{ such that} \\ & s_i \succeq_i s_i' \text{ for all } s_i' \in S_i, \\ & s_j \text{ is } \succeq_i\text{-nonnull for all } s_j \in S_j, \\ & s_j \text{ is } \mathcal{L}(\succeq_i)\text{-possible only if } s_j \in P_j^{n-1} \}. \end{aligned} \tag{17}$$

This iterative procedure delivers (in a finite number of rounds, due to the finiteness of $S_i$) the largest $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set.

**Proposition 4.** *The largest $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set $P_1^* \times P_2^*$ is nonempty, and is given by $P_1^* \times P_2^* = \cap_{n=0}^{\infty} P_1^n \times \cap_{n=0}^{\infty} P_2^n$.*

*Permissibility* (Brandenburger, 1992, Definition 2, p. 286) can be seen as an instance of $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility as follows. Parallel to Eq. (8), for any $\succeq_i \in \mathcal{M}(S_j)$, let

$$\mathcal{L}(\succeq_i) = \{ \sigma_j \subseteq S_j | c\sigma_j d \succ_i d\sigma_j c' \text{ for all } c', c, d \in \mathbb{R} \text{ such that } c' > c > d \} \tag{18}$$

be the collection of likely events in $S_j$. Let $\mathcal{M}(S_j)$ be the set of all lexicographic expected utility preference relations, with an identical continuous strictly increasing vNM index. Then $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility is permissibility. The Dekel-Fudenburg procedure (namely, elimination of inadmissible strategies, followed by iterated elimination of strictly dominated strategies) delivers the largest permissible set (Brandenburger, Proposition 2, p. 287).

Another example of $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility is as follows.[10] Parallel to Eq. (4) of Section 2, suppose that $\mathcal{M}(S_j)$ is the set of all preference relations representable by

$$U_{S_j}(f) = \sum_{\sigma_j \in 2^{S_j} \setminus \{\emptyset\}} b_i(\sigma_j) \min_{s_j \in \sigma_j} u_i(f(s_j)) \quad \forall f \in \mathcal{F}(S_j), \tag{19}$$

where the vNM index $u_i$ is fixed, but the basic probability assignment $b_i$ is variable. Parallel to Eq. (6), for any $b_i$, define

$$\text{supp } b_i = \{ \sigma_j \in 2^{S_j} \setminus \{\emptyset\} | b_i(\sigma_j) > 0 \}.$$

By Eq. (7), for any $\succeq_i \in \mathcal{M}(S_j)$ with corresponding $b_i$, $s_j$ is $\succeq_i$-nonnull for all $s_j \in S_j$ if and only if $\bigcup_{\sigma_j \in \text{supp } b_i} \sigma_j = S_j$. Parallel to Eq. (10), define

$$\min \text{supp } b_i = \{ \sigma_j \in 2^{S_j} \setminus \{\emptyset\} | b_i(\sigma_j) > 0, \text{and for all } \sigma_j' \subset \sigma_j, b_i(\sigma_j') = 0 \}.$$

Suppose we also adopt Eq. (18) as the collection of likely events in $S_j$. Then Eq. (9) tells us that the set of $\mathcal{L}(\succeq_i)$-possible states in $S_j$ is equal to $\bigcup_{\sigma_j \in \min \text{supp } b_i} \sigma_j$. Finally, for any

[10]Similar to this example, Mukerji's (1995) solution concept for the $\epsilon$-contamination model, as simplified in Epstein (1997, footnote 5, p. 15), can also be stated as a special case of $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility.

$s_i \in S_i$, and any nonempty $\Sigma_j \subseteq 2^{S_j} \setminus \{\emptyset\}$, say that $s_i$ is *weakly dominated given* $\Sigma_j$ if there exists a probability measure $p_i$ on $S_i$ such that for every $\sigma_j \in \Sigma_j$,

$$\sum_{s_i' \in S_i} p_i(s_i') \min_{s_j \in \sigma_j} u_i(g_i(s_i', s_j)) \geq \min_{s_j \in \sigma_j} u_i(g_i(s_i, s_j)),$$

and for at least one $\sigma_j \in \Sigma_j$,

$$\sum_{s_i' \in S_i} p_i(s_i') \min_{s_j \in \sigma_j} u_i(g_i(s_i', s_j)) > \min_{s_j \in \sigma_j} u_i(g_i(s_i, s_j)).$$

Pearce's (1984, p. 1049) Lemma 4 can be easily modified to establish the equivalence of the following two statements: (i) $s_i$ is not weakly dominated given $\Sigma_j$; (ii) there exists $\succeq_i \in \mathcal{M}(S_j)$ with corresponding $b_i$ such that supp $b_i = \Sigma_j$, and $s_i \succeq_i s_i'$ for all $s_i' \in S_i$. Putting everything together, Eq. (17) becomes

$$\begin{aligned} P_i^n = \{s_i \in S_i | \exists \Sigma_j \subseteq 2^{S_j} \setminus \{\emptyset\} & \text{ such that} \\ & s_i \text{ is not weakly dominated given } \Sigma_j, \\ & \cup_{\sigma_j \in \Sigma_j} \sigma_j = S_j, \\ & \cup_{\sigma_j \in \min \Sigma_j} \sigma_j \subseteq P_j^{n-1}\}, \end{aligned} \tag{20}$$

where

$$\min \Sigma_j \equiv \{\sigma_j \in \Sigma_j | \text{For all } \sigma_j' \subset \sigma_j, \sigma_j' \notin \Sigma_j\}$$

is the collection of minimal events in $\Sigma_j$. Eq. (20) says that for every positive integer $n$, $s_i$ belongs to $P_i^n$ if there exists $\Sigma_j$ such that the following three conditions are satisfied: (i) $s_i$ is not weakly dominated given $\Sigma_j$; (ii) $\Sigma_j$ is a covering of $S_j$; (iii) $\min \Sigma_j$ is a covering of at most $P_j^{n-1}$. Note that (ii) and (iii) do not run into conflict precisely because events in $\Sigma_j$ can have cardinality bigger than one, so that $\min \Sigma_j$ can be a proper subset of $\Sigma_j$. To stress the significance of this flexibility, let $R_i^0 = S_i$, and for each positive integer $n$, let

$$\begin{aligned} R_i^n = \{s_i \in S_i | \exists \Sigma_j \subseteq \{\{s_j\}\}_{s_j \in S_j} & \text{ such that} \\ & s_i \text{ is not weakly dominated given } \Sigma_j, \\ & \cup_{\sigma_j \in \min \Sigma_j} \sigma_j \subseteq R_j^{n-1}\}. \end{aligned} \tag{21}$$

By Pearce's (1984, p. 1048) Lemma 3, $R_i^n$ is the set of strategies surviving $n$ rounds of iterated elimination of strictly dominated strategies. Note that every event in $\Sigma_j$ is required to be a singleton, and hence the condition "$\cup_{\sigma_j \in \Sigma_j} \sigma_j = S_j$" cannot be included in Eq. (21).

We illustrate Eq. (20) using the games depicted in Figs. 1 and 2. (In both figures, player 1 chooses the row, and player 2 chooses the column; payoffs are in terms of vNM utilities.) Let us first consider Fig. 1. It is obvious that $D$ is weakly dominated given any covering of $S_2$; but neither $U$ nor $M$ is weakly dominated given the covering $\Sigma_2 = \{\{L\}, S_2\}$. As for player 2's strategies, $R$ is weakly dominated given any covering of $S_1$; but $L$ is not weakly dominated given the covering $\Sigma_1 = \{\{U, M\}, S_1\}$. So $P_1^1 \times P_2^1 = \{U, M\} \times \{L\}$. Note that the unique event $\{U, M\}$ in $\min \Sigma_1$ just covers $P_1^1$, and the unique event $\{L\}$ in $\min \Sigma_2$ just covers $P_2^1$. Hence the iterative procedure stops, with $P_1^* \times P_2^* = P_1^1 \times P_2^1 = \{U, M\} \times \{L\}$.[11]

| | $L$ | $R$ |
|---|---|---|
| $U$ | $2, 2$ | $2, 2$ |
| $M$ | $3, 1$ | $0, 1$ |
| $D$ | $-1, 0$ | $-1, -1$ |

Figure 1: A strategic game

Turn to the game in Figure 2. Neither $U$ nor $D$ is weakly dominated given the covering $\{\{L\}, \{C\}, \{R\}\}$ of $S_2$. Obviously, $L$ is not, but both $C$ and $R$ are, weakly dominated given any covering of $S_1$. So $P_1^1 \times P_2^1 = \{U, D\} \times \{L\}$, and in the next round, we are only allowed to consider any covering $\Sigma_2$ of $S_2$ with the property that $\min \Sigma_2$ contains only $\{L\}$; but $U$ is weakly dominated given any such covering. Hence $P_1^* \times P_2^* = P_1^2 \times P_2^2 = \{D\} \times \{L\}$.[12]

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $0, 1$ | $1, 0$ | $-2, 0$ |
| $D$ | $0, 1$ | $0, 0$ | $4, 0$ |

Figure 2: A strategic game

Our final task is to establish the foundation of $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility. Let $T_1 \times T_2$ be a finite type space, with typical type profile $(t_1, t_2)$. For convenience, we will frequently refer to player $i$ with type $t_i$ as "player $t_i$." Player $t_i$ knows his own actual type; however, since he may not know player $j$'s type, he regards $T_j$ as the grand world. Let $\succeq^{t_i}$ be $t_i$'s preference relation on $\mathcal{F}(T_j)$, and $\mathbf{s}_i(t_i) \in S_i$ be the strategy chosen by $t_i$ in the game. His preference relation $\succeq^{t_i}_{S_j}$ on $\mathcal{F}(S_j)$ is derived from $\succeq^{t_i}$ along the line of Eq. (1); to be precise, for all $f, f' \in \mathcal{F}(S_j)$,

$$f \succeq^{t_i}_{S_j} f' \quad \text{if} \quad \tilde{f} \succeq^{t_i} \tilde{f}', \tag{22}$$

where $\tilde{f}(t_j) = f(\mathbf{s}_j(t_j))$ and $\tilde{f}'(t_j) = f'(\mathbf{s}_j(t_j))$ for all $t_j \in T_j$. Assume that $\succeq^{t_i}_{S_j} \in \mathcal{M}(S_j)$ for all $t_i \in T_i$; in words, $\succeq^{t_i}_{S_j}$ conforms to the model of preference $\mathcal{M}$.

Recall that any strategy $s_i \in S_i$ can be interpreted as an element of $\mathcal{F}(S_j)$. Similarly, $s_i$ can also be interpreted as an element of $\mathcal{F}(T_j)$ in the following sense: For each $t_j \in T_j$,

[11] Iterated strict dominance delivers $\{U, M\} \times \{L, R\}$. Both iterated admissibility and the Dekel-Fudenburg procedure deliver $\{M\} \times \{L\}$, which is also the only self-admissible set (in the sense of Brandenburger et al., 2008, Definition 3.3, p. 320).

[12] Iterated strict dominance, iterated admissibility, and the Dekel-Fudenburg procedure all deliver $\{U, D\} \times \{L\}$, which is also the largest self-admissible set.

$s_i$ specifies the consequence $g_i(s_i, \mathbf{s}_j(t_j))$. So it is meaningful to say that $t_i$ is *rational* if $\mathbf{s}_i(t_i) \succeq^{t_i} s_i$ for all $s_i \in S_i$; in words, $t_i$ is rational if he chooses an optimal strategy.

Intuitively, player $t_i$ is cautious if he does not completely rule out any strategy of his opponent. Formally, say that $t_i$ is *cautious* if for each $s_j \in S_j$, $\mathbf{s}_j^{-1}(s_j) \equiv \{t_j \in T_j | \mathbf{s}_j(t_j) = s_j\}$ is a $\succeq^{t_i}$-nonnull event. Clearly, if $t_i$ is cautious, then $\mathbf{s}_j^{-1}(s_j)$ is nonempty for all $s_j \in S_j$, and $\{\mathbf{s}_j^{-1}(s_j)\}_{s_j \in S_j}$ is a partition of $T_j$; consequently, $S_j$ is in effect a small world induced from $T_j$.

Given $\mathcal{L}$, $\mathcal{L}(\succeq^{t_i}) \subseteq 2^{T_j}$ is the collection of likely events (satisfying L1–L4 of Section 2, with $\succeq^{t_i}$ in place of $\succeq_S$). For any event $\tau_j \subseteq T_j$, say that player $t_i$ *$\mathcal{L}(\succeq^{t_i})$-believes* $\tau_j$ if every $\mathcal{L}(\succeq^{t_i})$-possible state is contained in $\tau_j$.[13] Define

$$T_i^1 = \{t_i \in T_i | t_i \text{ is rational and cautious}\}, \tag{23}$$

and recursively define, for every positive integer $n$,

$$T_i^{n+1} = \left\{t_i \in T_i^n | t_i\ \mathcal{L}(\succeq^{t_i})\text{-believes } T_j^n\right\}. \tag{24}$$

Define $T_i^\infty = \cap_{n=1}^\infty T_i^n$. Eqs. (23) and (24) justify the interpretation of $T_1^\infty \times T_2^\infty$ as the set of all rational and cautious types, who also *commonly* *$\mathcal{L}(\succeq^{t_i})$-believe* that they are rational and cautious. So

$$S_i(T_i^\infty) \equiv \{s_i \in S_i | \text{There exists } t_i \in T_i^\infty \text{ such that } \mathbf{s}_i(t_i) = s_i\} \tag{25}$$

is the set of player $i$'s strategies that are consistent with rationality, caution, and common $\mathcal{L}(\succeq^{t_i})$-belief of rationality and caution. Combining this interpretation of $S_i(T_i^\infty)$ with Proposition 5 below, we obtain the epistemic conditions for $\langle \mathcal{L}, \mathcal{M} \rangle$-permissibility.[14]

**Proposition 5.** *For any $T_1 \times T_2$, $S_1(T_1^\infty) \times S_2(T_2^\infty)$ is a $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set. There exists $T_1 \times T_2$ such that $S_1(T_1^\infty) \times S_2(T_2^\infty)$ is the largest $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set.*

For Proposition 5 to go through, we need a possible small-world state to contain a possible grand-world state, and we need a small-world state containing a nonnull grand-world state to be nonull; that is why Propositions 2 and 3 of Section 3 are crucial.

# Appendix

### Proof of Proposition 2

By Proposition 1, Proposition 2 can be restated as follows: *For every $\sigma \in \min \mathcal{L}(\succeq_S)$, and every $s \in \sigma$, there exists $t \in \mathbf{s}^{-1}(s)$ such that $t$ is $\mathcal{L}(\succeq)$-possible.* To prove this statement, we assume the contrary. That is, suppose we are able to fix $\sigma \in \min \mathcal{L}(\succeq_S)$ and $s \in \sigma$ such that every $t \in \mathbf{s}^{-1}(s)$ is not $\mathcal{L}(\succeq)$-possible. Property L5 and $\sigma \in \min \mathcal{L}(\succeq_S)$ imply $\mathbf{s}^{-1}(\sigma) \in \mathcal{L}(\succeq)$, which in turn, by definition, imply that

$$\text{there exists } \tau \in \min \mathcal{L}(\succeq) \text{ such that } \tau \subseteq \mathbf{s}^{-1}(\sigma). \tag{26}$$

[13]Suppose that $\mathcal{L}$ is also a definition of believed events (as formalized in Section 3). Then player $t_i$ $\mathcal{L}(\succeq^{t_i})$-believes $\tau_j$ if and only if $\tau_j \in \mathcal{L}(\succeq^{t_i})$.

[14]Proposition 5, when specialized to lexicographic expected utility preferences, is comparable to the result in Brandenburger et al. (2008, Section 11B, pp. 332–333), where $T_i$ is allowed to be infinite.

By Proposition 1, the hypothesis that every $t \in \mathbf{s}^{-1}(s)$ is not $\mathcal{L}(\succeq)$-possible can be rewritten as

$$\text{for all } \tau \in \min \mathcal{L}(\succeq), \mathbf{s}^{-1}(s) \cap \tau = \emptyset. \tag{27}$$

Eqs. (26) and (27) imply that there exists $\tau \in \min \mathcal{L}(\succeq)$ such that $\tau \subseteq \mathbf{s}^{-1}(\sigma) \setminus \mathbf{s}^{-1}(s) = \mathbf{s}^{-1}(\sigma \setminus \{s\})$, and so by L1, $\mathbf{s}^{-1}(\sigma \setminus \{s\}) \in \mathcal{L}(\succeq)$. Property L5 and $\mathbf{s}^{-1}(\sigma \setminus \{s\}) \in \mathcal{L}(\succeq)$ imply that $\sigma \setminus \{s\} \in \mathcal{L}(\succeq_S)$, contradicting $s \in \sigma \in \min \mathcal{L}(\succeq_S)$.

**Proof of Proposition 3**

Suppose that statement (ii) of Proposition 3 is not valid. Note that, by L5, if B1 (P1, respectively) holds for $\mathcal{L}(\succeq)$, then B1 (P1, respectively) holds for every $\mathcal{L}(\succeq_S)$. So invalidity of statement (ii) implies that neither Eq. (11) nor Eq. (12) holds for the finest small world. Consequently, we are able to fix $\tau \in \min \mathcal{L}(\succeq)$ with cardinality $|\tau| > 1$, fix $\tau' \in \min \mathcal{L}(\succeq)$ such that $\tau' \neq \tau$, and fix $t \in \tau \setminus \tau'$. Since $t \in \tau \in \min \mathcal{L}(\succeq)$ and $|\tau| > 1$, we have $\{t\} \notin \mathcal{L}(\succeq)$. Since $t \notin \tau'$, we have $\tau' \subseteq T \setminus \{t\}$. Property L1, $\tau' \in \min \mathcal{L}(\succeq)$, and $\tau' \subseteq T \setminus \{t\}$ imply that $T \setminus \{t\} \in \mathcal{L}(\succeq)$. Consider the small world $S = \{\{t\}, T \setminus \{t\}\}$. Property L5, $\{t\} \notin \mathcal{L}(\succeq)$, and $T \setminus \{t\} \in \mathcal{L}(\succeq)$ imply that $\{\{t\}\} \notin \mathcal{L}(\succeq_S)$ and $\{T \setminus \{t\}\} \in \mathcal{L}(\succeq_S)$; therefore, by Proposition 1, $\{t\}$ is not $\mathcal{L}(\succeq_S)$-possible. But Proposition 1 and $t \in \tau \in \min \mathcal{L}(\succeq)$ imply that $t$ is $\mathcal{L}(\succeq)$-possible. Hence statement (i) of Proposition 3 is not valid either.

Conversely, suppose that statement (ii) is valid. There are, of course, two cases to consider. First, suppose that $\mathcal{L}$ is a definition of believed events. By Eq. (11),

$$\min \mathcal{L}(\succeq) = \{\{t | t \text{ is } \mathcal{L}(\succeq)\text{-possible}\}\}. \tag{28}$$

Property L5 and Eq. (28) imply that for every small world $S$,

$$\min \mathcal{L}(\succeq_S) = \{\{s | \exists t \in \mathbf{s}^{-1}(s) \text{ such that } t \text{ is } \mathcal{L}(\succeq)\text{-possible}\}\}. \tag{29}$$

Proposition 1 and Eq. (29) imply the validity of statement (i). Second, suppose that $\mathcal{L}$ is a definition of possible events. By Eq. (12),

$$\min \mathcal{L}(\succeq) = \{\{t\} | t \text{ is } \mathcal{L}(\succeq)\text{-possible}\}. \tag{30}$$

Property L5 and Eq. (30) imply that for every small world $S$,

$$\min \mathcal{L}(\succeq_S) = \{\{s\} | \exists t \in \mathbf{s}^{-1}(s) \text{ such that } t \text{ is } \mathcal{L}(\succeq)\text{-possible}\}. \tag{31}$$

Proposition 1 and Eq. (31) imply the validity of statement (i).

**Proof of Proposition 4**

We prove below that every (hence the largest) $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set is a subset of $\cap_{n=0}^{\infty} P_1^n \times \cap_{n=0}^{\infty} P_2^n$, and $\cap_{n=0}^{\infty} P_1^n \times \cap_{n=0}^{\infty} P_2^n$ is a nonempty $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set.

Fix any $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set $P_1 \times P_2$. Obviously, $P_1 \times P_2 \subseteq P_1^0 \times P_2^0 \equiv S_1 \times S_2$. To prove that $P_1 \times P_2 \subseteq \cap_{n=0}^{\infty} P_1^n \times \cap_{n=0}^{\infty} P_2^n$, it suffices to prove by induction that for every positive integer $n$, if $P_1 \times P_2 \subseteq P_1^{n-1} \times P_2^{n-1}$, then $P_1 \times P_2 \subseteq P_1^n \times P_2^n$. Recall that, according to Definition 2, for each $s_i \in P_i$, there exists $\succeq_i \in \mathcal{M}(S_j)$ such that the following three conditions are satisfied:

(i) $s_i \succeq_i s_i'$ for all $s_i' \in S_i$.

(ii) $s_j$ is $\succeq_i$-nonnull for all $s_j \in S_j$.

(iii) $s_j$ is $\mathcal{L}(\succeq_i)$-possible only if $s_j \in P_j$.

Condition (iii) and $P_j \subseteq P_j^{n-1}$ together imply that $s_j$ is $\mathcal{L}(\succeq_i)$-possible only if $s_j \in P_j^{n-1}$, which can be combined with conditions (i) and (ii) to conclude that $s_i \in P_i^n$.

It is easy to see that for every positive integer $n$, $P_i^n$ is nonempty and $P_i^n \subseteq P_i^{n-1}$. So, due to the finiteness of $S_i$, there exists a positive integer $N$ such that $P_1^{N-1} \times P_2^{N-1} = P_1^N \times P_2^N = \cap_{n=0}^{\infty} P_1^n \times \cap_{n=0}^{\infty} P_2^n$, implying that the latter is a $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set.

**Proof of Proposition 5**

For any $s_i \in S_i(T_i^\infty)$, fix $t_i \in T_i^\infty$ with the property that $\mathbf{s}_i(t_i) = s_i$. By Eq. (25), at least one such $t_i$ exists. Recall the assumption that $\succeq_{S_j}^{t_i} \in \mathcal{M}(S_j)$. So $S_1(T_1^\infty) \times S_2(T_2^\infty)$ is a $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set if the following three conditions are satisfied:

(i) $s_i \succeq_{S_j}^{t_i} s_i'$ for all $s_i' \in S_i$.

(ii) $s_j$ is $\succeq_{S_j}^{t_i}$-nonnull for all $s_j \in S_j$.

(iii) $s_j$ is $\mathcal{L}(\succeq_{S_j}^{t_i})$-possible only if $s_j \in S_j(T_j^\infty)$.

Since $t_i \in T_i^\infty$, $t_i$ must be rational and cautious. Eq. (22) and rationality of $t_i$ imply that condition (i) above is satisfied. By Eq. (22), Proposition 3, and the fact that nonnull states can be derived from a definition of believed/possible events, for every $s_j \in S_j$,

$$\text{if there exists } t_j \in \mathbf{s}_j^{-1}(s_j) \text{ such that } t_j \text{ is } \succeq^{t_i}\text{-nonnull, then } s_j \text{ is } \succeq_{S_j}^{t_i}\text{-nonnull.} \tag{32}$$

As $t_i$ is cautious, we indeed have, for every $s_j \in S_j$, there exists $t_j \in \mathbf{s}_j^{-1}(s_j)$ such that $t_j$ is $\succeq^{t_i}$-nonnull. So, Eq. (32) and the fact that $t_i$ is cautious imply condition (ii). By Eq. (22) and Proposition 2, for every $s_j \in S_j$,

$$\text{if } s_j \text{ is } \mathcal{L}(\succeq_{S_j}^{t_i})\text{-possible, then there exists } t_j \in \mathbf{s}_j^{-1}(s_j) \text{ such that } t_j \text{ is } \mathcal{L}(\succeq^{t_i})\text{-possible.} \tag{33}$$

By Eq. (24) and the fact that $t_i \in T_i^\infty$, for every $t_j \in T_j$,

$$\text{if } t_j \text{ is } \mathcal{L}(\succeq^{t_i})\text{-possible, then } t_j \in T_j^\infty. \tag{34}$$

Eqs. (25), (33), and (34) imply that condition (iii) is also satisfied.

Finally, we construct $T_1 \times T_2$ such that $S_1(T_1^\infty) \times S_2(T_2^\infty)$ is the largest $\langle \mathcal{L}, \mathcal{M} \rangle$-permissible set $P_1^* \times P_2^*$. For any $s_i \in P_i^*$, fix a preference relation $\succeq_{S_j}^{s_i} \in \mathcal{M}(S_j)$ such that the following three conditions are satisfied:

(i) $s_i \succeq_{S_j}^{s_i} s_i'$ for all $s_i' \in S_i$.

(ii) $s_j$ is $\succeq_{S_j}^{s_i}$-nonnull for all $s_j \in S_j$.

(iii) $s_j$ is $\mathcal{L}(\succeq^{s_i}_{S_j})$-possible only if $s_j \in P^*_j$.

By Definition 2, at least one such $\succeq^{s_i}_{S_j}$ exists. As for any $s^c_i \in S_i \setminus P^*_i$, simply pick an arbitrary $s_i \in P^*_i$ and let $\succeq^{s^c_i}_{S_j} = \succeq^{s_i}_{S_j}$. Since $\succeq^{s_i}_{S_j}$ satisfies conditions (ii) and (iii) above, $s^c_i \succeq^{s_i}_{S_j} s'_i$ cannot hold for all $s'_i \in S_i$; otherwise $s^c_i$ would belong to $P^*_i$, a contradiction. Let $T_i = S_i$; for all $s_i \in T_i$, let $\succeq^{s_i} = \succeq^{s_i}_{S_j}$ and $\mathbf{s}_i(s_i) = s_i$. Then we immediately have $P^*_1 \times P^*_2 = T^1_1 \times T^1_2 = T^\infty_1 \times T^\infty_2 = S_1(T^\infty_1) \times S_2(T^\infty_2)$.

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