Financing Multi-Stage Projects Under Moral Hazard and Limited Commitment.

Josepa Miquel-Florensa*
York University

May 8, 2007

Abstract

We present the optimal contract for financing a project that has N stages to be completed sequentially when the principal cannot commit to abandoning the project before it is completed and the agent values the project to be completed

In a dynamic moral hazard setting, we find that the optimal contract provides decreasing transfers for successive unsuccessful attempts in a given stage, and smaller transfers when subsequent stages are reached. We find that the optimal sequence of transfers is greater when the larger the exogenous probability of returning to a preceding stage and the higher the principal's cost of stage verification. When the agent values the intermediate stages, we find that smaller transfers are optimal.

Keywords: Dynamic contracts, Moral Hazard, Foreign Aid, multi-stage projects.

JEL codes: D82, D86, F35, O12

" ... the donors wouldn't be donors if they didn't care for the poor in the recipient country, (...) Even if the conditions are not met, the donors want to alleviate the lot of the poor, and so they give the aid anyway. The recipients can anticipate this behavior of donors and thus sit tight without doing reforms or helping the poor, expecting to get the loans anyway" (W. Easterly, The Elusive Quest for Growth)

*York University, Department of Economics. E-mail: pepita@econ.yorku.ca. I would like to thank Atila Abdulkadiroglu, Charles Calomiris, Dirk Bergemann, Debraj Ray, Bernard Salanie, Xavier Sala-i-Martin, and seminar participants at Columbia University for useful comments and suggestions. All errors remain my own.

1 Introduction

How should Development Agencies finance multi-stage projects when they cannot observe the recipients' use of funds? How should a company provide appropriate incentives to departmental managers to complete multi-stage projects when these managers are specific for the firm? These examples have special characteristics of the principal and agent in common. The principal cannot commit to abandoning the contract before the project is completed, whatever the outcome of investment at each stage is. In addition, the agent obtains a reward from the project itself and values the attainment of the project.

To obtain the optimal contract in this setting, we propose a dynamic moral hazard model that accounts for the special characteristics of multi-stage projects and its principals and agents. The projects considered have a finite number of stages, with completion of each stage verifiable by the principal. The action of the agent, her investment at each trial at each stage, determines the "transition" of the relationship: success of the investment in one stage is what leads the project to the consecutive one. The possibility that, due to exogenous factors, the project can return to previous stages is also considered.

Each period, the principal transfers an amount of money to the agent, who decides how to allocate the funds between consumption and investment. The principal observes the stage the project is and the number of periods the project has been at this stage, but not the amount invested. The principal makes funds transfers conditional on his information about the progress of the project. Accordingly, a cost of checking the situation (progress) of the project at each period is introduced as an extension of the baseline model.

In this framework, a special characteristic of the principal is that he faces a Samaritan Dilemma[1]: whatever the outcome of the investment, the principal cannot commit to abandoning the relationship. In the literature, this problem is approached through conditional contracts. For example, in extant foreign aid literature, Dranzen (1999), Svenson (2000, 2003), and Azam and Laffont (2003), among others, present models that condition the aid flows on a given performance, a degree of political and economic change, or a given consumption level for the poorest people in the recipient country, respectively. As well, the time horizon of these models is either two periods or an infinite horizon. In practice, conditionality has not been enforced[2]. The contract we propose accounts for the commitment problem of the principal, assuming that the contract does not end until the project is completed, which makes the time horizon indeterminate.

The principal cannot commit to abandon the contract before the project is completed, but we assume that he can commit to a sequence of transfers for each possible history of success and failure on the successive stages of the project. We start presenting the optimal stationary contract, where the principal can only commit to different transfers for different stages of the project, and we continue with the contract where different transfers are allowed for successive trials on a given stage.

[1] Term introduced by Buchanan (1975)

[2] Killick (1998), Dreher (2002) and The World Bank (2005) present reviews of the literature and examples of time inconsistency involving conditional aid contracts.

The optimal contract prescribes different transfer schemes at specific stages of the project. At the first stage of the project, the optimal contract is stationary and provides the agent a lifetime utility that is greater than his reservation value. This is due to the fact that the agent values the project and also due to the non-separability of the consumption and effort-investment decision: only the existence of a positive probability that the valued project may be completed in the future makes the agent better off as a result of the contract.

At intermediate stages, the principal has more room to choose a sequence of transfers after failure: he can credibly threaten the agent with smaller transfers for successive failures at a given intermediate stage, so long as he provides him with at least as much utility as he had before the start of the relationship. In this situation, the optimal contract prescribes decreasing transfers as successive unsuccessful trials appear in a given stage and smaller transfers once the next stage is reached. At the last stage of the project, the principal does not choose the promised utility in case of success. Transfers also decrease for successive unsuccessful trials at the last stage until the project is entirely complete. In both types of stages, the sequence of transfers converges to the optimal stationary contract.

This model has the ability to work as a "cookbook" of sorts for donor-recipient contracts: given the characteristics of the agent, the principal and the project, it prescribes the optimal contract. Comparative static exercises give the "cookbook" its rules: greater funds transfers are given for greater probabilities of going backwards in the project. Also, greater funds transfers are given the higher the principal's cost of checking the situation of the project at each stage/trial. And smaller transfers are provided the more the agent values each intermediate stage.

The structure of the paper is as follows. In Section 2, we present the structure of the model. In Section 3, we present the optimal stationary contract. Section 4 describes the optimal contract for each stage of the project when transfers are allowed to vary for successive trials in each stage. Section 5 presents three extensions to the model: the introduction of a probability of falling to preceding stages, or a cost of state verification and intermediate stages valued by the agent. Section 6 concludes.

2 Structure of the model

The objective of the proposed contract is the completion of a project by the recipient. We consider projects with N sequential stages that are verifiable by the donor. The value for the agent of the completed project is W. We start with the assumption that the completion of intermediate stages does not provide any utility to the agent, that he only values the fully completed project. We relax this assumption on the extensions of the model, where the agent values some intermediate stages of the project as well.

The transition among stages works as follows. The probability of getting to the next stage is given by an increasing, concave and differentiable function $p(.)$ of the investment performed by the agent. There is a minimum level of investment $\bar{\imath}$ for the project to have a positive probability of success. We start with the assumption that when one stage is reached, it cannot be destroyed, and in the extensions, we allow for an exogenous probability (denoted by α) of returning to the preceding stages.

Let τ_t denote transfer in period t, and i_t denote period t investment. Figure 1 shows the game tree representation of the game when there exists an exogenous probability α of falling to the preceding stage.

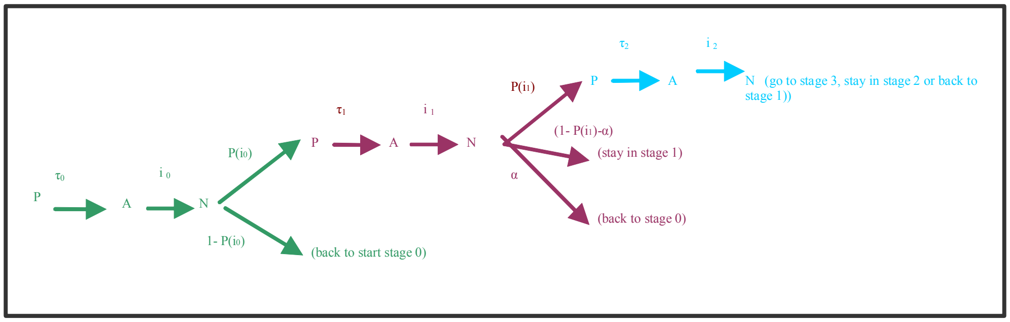

Figure 1

The **principal** has information about the stage that the project is in and the number of periods it has been in this stage. He offers a contract that specifies a sequence of funds transfers after each possible history of successes and failures of the investment at each of the stages.

Let h_t be the history of success and failure of the successive stages of the project up to period t. The principal's strategy is given by $\tau(h_t)$, transfer after each possible observed history of the project h_t. The transition probability from h_t to $h_{t+1} \epsilon H_{t+1} \mid h_t$, where $H_{t+1} \mid h_t$ denotes the set of all possible histories at period t+1 given history up to t, is determined by the investment the agent performs with the transfer received, $p(h_{t+1} \mid h_t, i(h_t))$.

We consider a principal that faces a *Samaritan Dilemma*: whatever the outcome of the investment, at any trial and at any stage, the principal cannot commit to abandoning the relationship with the agent before the project is completed. This commitment problem introduces an additional constraint on the contract, since no cancellation clauses would be credible. For this reason, we assume that the contract has an indeterminate time horizon: it does not end until the project is completed.

Given the commitment problem of the principal, it is natural to consider two possible scenarios: either the principal is able to commit to a sequence of transfers conditional on the history of success and failures on the successive trials, or he is only able to commit to different transfers as successive stages are reached.

Each period, the **agent** chooses from the transferred funds that he gets the amount he wants to invest and consume. It is assumed that the agent does not have any other source of income available to build the project other than the aid received, and that no savings from the received aid are allowed[3].

[3]This assumption is made to avoid adverse selection problems due to the non-observable savings as the project

Let $u(.)$ be the increasing, concave and differentiable instantaneous utility function from consumption of the agent. We denote by W the lifetime utility the completed project provides to the agent, and by $w = W(1-\beta)$ the instantaneous utility flow the project provides to the agent.

Define

$$V(\tau(h_t), i(h_t)) = \sum_{s=t}^{\infty} \beta^{s-1} u(\tau(h_s) - i(h_s)) p(h_s \mid h_{s-1}) \tag{1}$$

$$C(\tau(h_t), i(h_t)) = \sum_{s=t}^{\infty} \beta^{s-1} \tau(h_s) p(h_s \mid h_{s-1}) \tag{2}$$

as the present discounted value for the agent and the expected cost for the principal of the strategies (τ, i) for the subgame starting after history h_t.

We follow Spear and Srivastava (1987) and present the problem in the recursive form: at each possible history h_t where project is at stage n, the principal chooses the triplet (τ, V', V_{n+1}): transfer, τ, promised utility in case investment at this stage is successful, V_{n+1}, and promised utility in case of failure, V'.

The optimal contract needs to satisfy the usual promise keeping constraint, to ensure consistency of the recursive formulation, and the incentive compatibility constraint. Moreover, participation and minimum investment constraints need to be satisfied: we need to ensure that the agent participates in the contract, and that when in he provides at least the minimum investment required to have a positive probability of success.

To be willing to participate in the contract, the agent should get at least his reservation utility V_0, and we need to make sure that the agent is willing to invest at least $\bar{\imath}$ at each stage. At the last stage of the project, the minimum level of utility that can be provided to the agent so that he invests $\bar{\imath}$ in a stationary contract is given by

$$V_{\min}^{N} = \frac{u(\tilde{\tau} - \bar{\imath}) + \beta p(\bar{\imath}) W}{1 - \beta(1 - p(\bar{\imath}))} \geqslant V_0 \tag{3}$$

where $\tilde{\tau}$ is the minimum incentive compatible transfer that induces the minimum investment, and at last stage is given by

$$g(\tilde{\tau}) = u'(\tilde{\tau} - \bar{\imath})(1 - \beta(1 - p(\bar{\imath}))) - \beta p'(i)(W(1-\beta) - u(\tilde{\tau} - \bar{\imath})) = 0$$

Working backwards we find the expressions for $V_{\min}^{n}$ at each stage n. These give the minimum utilities to be provided to the agent in each stage so that she performs the minimum investment.

The contracts that satisfy all constraints have a special characteristic: utility provided to the agent at each stage is greater than her reservation utility. This **"participation bonus"** is due to the fact that a positive probability of the valued project being completed makes the agent already "better off" at the moment the contract is signed.

evolves. Werning (2000, 2002) approach can not be applied due to the nonseparability of investment and efford decision.

2.1 Agent's problem

The agent chooses investment given the contract she is offered. Using the recursive formulation from [1] , at the n^{th} stage[4], the agent's problem is

$$V_n = \max_i \; u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right]$$

where $u(.)$ is the instantaneous utility function of the agent, i investment, and β the agent's discount factor. We assume $u(0) = 0, u'(0) > 0$.

First order (necessary and sufficient) condition of the agent's problem determines her investment choice,

$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V') \tag{4}$$

Claim 1 *Investment is increasing with transfer received. Investment is increasing with promise in case of success and decreasing with promise in case of failure.*

Proof. From the first order condition of the agent's problem, we get that

$$\begin{aligned}
\frac{di}{d\tau} &= -\frac{u''(\tau - i)}{-u''(\tau - i) - \beta p''(i)(V_{n+1} - V')} > 0 \\
\frac{di}{dV'} &= -\frac{\beta p'(i)}{-u''(\tau - i) - \beta p''(i)(V_{n+1} - V')} < 0 \\
\frac{di}{dV_{n+1}} &= -\frac{-\beta p'(i)}{-u''(\tau - i) - \beta p''(i)(V_{n+1} - V')} > 0
\end{aligned}$$

■

2.2 Principal's problem

The principal's objective is to minimize the cost of the contract that induces the agent to provide at least the minimum investment required to have a positive probability of success from the day the contract starts. As defined in [2] , cost is given by the expected present discounted value of the transfers established in the contract.

Let $C_n^*(V)$ be the expected cost for the principal of the optimal contract that in stage n of the project provides the agent a lifetime utility of V and induces the agent to invest at least the minimum investment to have a positive probability of success. Using the recursive formulation, the principal's problem at stage n is given by:

[4] For $1 < n \leqslant N$. For n=N-1, at the last stage of the project, $V_{N+1} = W$.

$$C_n^*(V) = \min_{\tau, V', V_{n+1}} \tau + \beta \left[p(i) C_{n+1}^*(V_{n+1}) + (1 - p(i)) C_n^*(V') \right]$$

$$s.t. V = u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right] \tag{5a}$$

$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V\prime) \tag{5b}$$

$$V' \geqslant V_{\min}^n, V_{n+1} \geqslant V_{\min}^{n+1} \tag{5c}$$

Where the vector $\{C^*\}$ is the fixed point of the T operator

$$TC_n(V) = \min_{\tau, V', V_{n+1}} \tau + \beta \left[p(i) C_{n+1}(V_{n+1}) + (1 - p(i)) C_n(V') \right]$$

$$s.t. V = u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right]$$

$$u'(\tau - i) = \beta \left[p'(i)(V_{n+1} - V') \right]$$

$$V' \geqslant V_{\min}^n, V_{n+1} \geqslant V_{\min}^{n+1}$$

And $[5a]$, $[5b]$ and $[5c]$ are the promise keeping, incentive compatible and minimum investment constraints. Cost function is increasing and concave, proof provided at Appendix A.

3 Stationary contract

The principal we are considering has a peculiar commitment problem: he cannot commit to abandon the contract before the project is accomplished. It is natural to consider, under this assumption, two possible scenarios: either the principal is able to commit to a sequence of transfers conditional on the history of success and failures on the successive trials, or he is only able to commit to different transfers as successive stages are reached.

We start by presenting the stationary contract, where the principal offers same transfer for all trials in a given stage until this stage is completed. We solve for the optimal stationary contract recursively: we calculate optimal transfer at last stage and we use the expected cost at last stage to calculate backwards the cost function for the preceding stages.

The agent's problem at last stage is[5]:

$$V_{N-1} = \max_i \; u(\tau_{N-1} - i) + \beta \left[p(i) W + (1 - p(i)) V_{N-1} \right]$$

and at the optimal investment choice i^*,

$$V_{N-1} = \frac{u(\tau_{N-1} - i^*) + \beta p(i^*) W}{\left[1 - p(i^*) \right] \beta}$$

[5] For an intermediate stage n, W becomes V_{n+1} and V_{N-1} is V_n, utility provided at that stage.

Investment is increasing with the size of the transfer at a decreasing rate and increasing with the value of the project to be completed (value of the continuation of the contract once stage is completed for intermediate stages).

The principal minimizes expected cost at each stage, denoted at stage n by $C_n(\tau_n)$. The cost at the n^{th} stage is given by the increasing and convex function

$$C_n(\tau_n) = \tau_n + \beta\left[p(i_n)C_{n+1}(\tau_{n+1}) + (1-p(i))C_n(\tau_n)\right] \quad (6)$$
$$C_n(\tau_n) = \frac{\tau_n + \beta p(i)C_{n+1}(\tau_{n+1})}{1-\beta(1-p(i))} \quad (7)$$

Rearranging terms we get

$$C_{n+1}(\tau_{n+1}) - C_n(\tau_n) = \frac{(1-\beta)C_n(\tau_n) - \tau_n}{\beta p(i)}$$

Given that τ_n is chosen optimally,

$$C_n(\tau_n) < \frac{\tau_n}{(1-\beta)}$$

since otherwise he could offer a contract with transfer τ_n forever whatever the outcome of investment is and would be cheaper and investment would be performed since it is how optimal τ_n is chosen. We get that cost is decreasing as stages are completed, $C_{n+1}(\tau_{n+1}) < C_n(\tau_n)$.

The solution to the minimization problem gives us the relation among the optimal transfers in successive stages of the project.

Proposition 1 *The optimal transfers are decreasing for the successive stages of the project.*

Proof. *The first order conditions of the cost minimization problem for stages n and (n+1) are:*

$$f(\tau_n) = \left[1+\beta p'(i_n)i'_\tau C^*_{n+1}(\tau_{n+1})\right](1-\beta) + \beta\left[p(\tau_n) - p'(i_n)i'_\tau.\tau_n\right]$$
$$f(\tau_{n+1}) = \left[1+\beta p'(i_{n+1})i'_\tau C^*_{n+2}(\tau_{n+2})\right](1-\beta) + \beta\left[p(\tau_{n+1}) - p'(i_{n+1})i'_\tau.\tau_{n+1}\right]$$

f(.) is an increasing and convex function, since $C^*_{n+1}(\tau_{n+1}) > C^*_{n+2}(\tau_{n+2}), \tau_{n+1} < \tau_n$ ■

4 Optimal contract for each stage of the project

The optimal contract for each stage (n) is characterized recursively by the triplet (τ, V'_{n+1}, V'): transfer, promised utility in case of success and promised utility in case of failure of the performed investment. We can distinguish three different types of stages according to the elements of this triplet chosen by the principal in each situation: first, intermediate and last stages.

4.1 Last Stage:

At the last stage, whenever investment succeeds the project is completed and the contract ends. Last stage contract is given by the pair (τ, V'), since promise in case of success is the value for the agent of the completed project. The principal's problem has the form:

$$C_N^*(V) = \min_{\tau, V'} \tau + \beta\left[(1-p(i))C_N^*(V')\right]$$

$$s.t. V = u(\tau - i) + \beta\left[p(i)W + (1-p(i))V'\right] \quad (\mu) \tag{8a}$$

$$u'(\tau - i) = \beta\left[p'(i)(W - V')\right] \quad (\lambda) \tag{8b}$$

$$V' \geqslant V_{\min}^N \quad (\gamma) \tag{8c}$$

$$(\tau \geqslant \bar{\imath}) \tag{8d}$$

where $C_N^*(V)$ is an increasing, concave and differentiable function that denotes the expected cost for the principal of the feasible, incentive compatible [8b] and promise keeping [8a] contract that provides the agent a utility V. The first order conditions of the principal's problem are:

$$\begin{aligned}
FOCw.r.t.\tau &: \quad 1 - \lambda u'(\tau - i) + \mu u''(\tau - i) = 0 \\
FOCw.r.t.V' &: \quad \beta(1-p(i))\frac{dC_N^*(V')}{dV'} + \beta p'(i)\mu - \beta(1-p(i))\lambda - \gamma = 0 \\
Envelope &: \quad \frac{dC_N^*(V)}{dV} = \lambda \geqslant 0
\end{aligned}$$

When $\gamma = 0$, and $V' \geqslant V_{\min}$ does not bind, rearranging the first order conditions we get:

$$\beta(1-p(i))\left(\frac{dC_N^*(V')}{dV'} - \frac{dC_N^*(V)}{dV}\right) = -\beta p'(i)\mu \leqslant 0$$

Since $C_N^*(V)$ is an increasing and convex function, $V_{\min}^N \leqslant V' \leqslant V$

When $\gamma > 0$, or $V' \geqslant V_{\min}$ binds we have a stationary contract from first trial on at $V = V' = V_{\min}^N$

Proposition 2 (Optimal contract last stage) *At the last stage of the project, the optimal contract offers a decreasing sequence of promised utilities in case of successive unsuccessful trials. This sequence converges to $V_{\min}^N$. Once $V_{\min}^N$ is reached, sequence of transfers is constant for the successive trials up to success.*

Optimal contract offers a decreasing sequence of transfers for the successive unsuccessful trials at the last stage while minimum level of utility at this stage is not reached. Once $V = V_{\min}^N$ is reached, the optimal contract offers a constant transfer for all trials at last stage, that is equal to the cheapest feasible transfer in a stationary contract.

Proof. To derive the optimal contract, we reduce the principal's problem to choose, for each level of utility V to be provided, $V'(V)$, promised utility in case of failure. We do this plugging the incentive compatibility [8b] and promise keeping [8a] constraints into the cost function. Once $V'(V)$ is chosen, [8b] and [8a] constraints give us $\tau(V)$.

Let $\tau(V)$ and $V'(V)$ be the optimal contract at last stage for a given level of promised utility V. We want to show that $V'(V) < V$ for all $V > V_{\min}^N$, that the sequence of promised utilities that our recursive contract provides for the successive failures at the last stage is decreasing.

We proceed by contradiction.

Step 1: Let $V'(V) = V > V^N_{\min}$ be the optimal contract with cost $C(V)$. We can propose an alternative contract $\hat{\tau}(V) = \tau(V) + \delta$ and $\hat{V}'(V) = V'(V) - \varepsilon$ such that it provides the agent at least utility V. Let $\hat{C}(V)$ be the cost of the new contract. Change in agent's utility is given by:

$$\begin{aligned} dV &= 0 = -\beta(1-p(i))\varepsilon + u'(\tau - i)\delta \\ \delta &= \frac{\beta(1-p(i))}{u'(\tau - i)}\varepsilon \end{aligned}$$

The change in cost from applying this new contract is

$$\begin{aligned} \hat{C}(V) - C(V) &= d\tau - \beta(1-p(i))\frac{dC_n(V')}{dV'}\varepsilon - \beta p'(i)\left(\frac{di}{d\tau}d\tau + \frac{di}{dV'}dV'\right)C(V') = \\ &= \beta p'(i)\frac{di}{dV'}\varepsilon C(V') < 0 \end{aligned}$$

What contradicts $V'(V) = V > V^N_{\min}$ to be optimal.

Step 2: $V'(V) > V > V^N_{\min}$ is not optimal. The proof is parallel to the previous step for the stationary contract at $V > V^N_{\min}$. We can find a new contract $\hat{\tau}(V)$ and $\hat{V}'(V)$ such that the promise keeping constraint is satisfied and is cheaper. The change in cost, given that the cost function is increasing and convex, is greater in this case. This contradicts $V'(V) > V$ to be optimal.

Step 3: We also want to show that this decreasing sequence of promised utilities converges to $V^N_{\min}$. Given that cost function is increasing and convex, and sequence of promised utilities is decreasing for $V > V^N_{\min}$, we have that sequence of V' is a Cauchy sequence, so is convergent. It has to converge to $\bar{V} \geqslant V^N_{\min}$, since it is not feasible that converges to $\bar{V} < V^N_{\min}$ given the constraints of the problem. At $V = V' = \bar{V} > V^N_{\min}$, we can propose an alternative to the stationary contract with $V' = \bar{V} - \varepsilon \geqslant V^N_{\min}$ that is cheaper, what leads to a contradiction. So $\bar{V} = V^N_{\min}$, sequence of promised utilities converges to the minimum feasible utility on a stationary contract.

Once $V'(V)$ is chosen, transfers are determined so that promise keeping constraint is satisfied. We know that promised utilities in case of failure are decreasing with the level of utility to be provided. Promise keeping constraint has the form

$$V = u(\tau - i) + \beta\left[p(i)W + (1-p(i))V'\right]$$

The change in level of utility to be provided for the next trial in this stage is given by the difference between the level provided and promise in case of failure at last trial. From the convexity of the cost function, we know that marginal cost of promised utilities is increasing with the level to be provided, so optimal changes of promises in case of failure are decreasing as trials in a given stage go on. Change on utility to be provided is given by:

$$dV = u'(\tau - i)d\tau + \beta(1-p(i))dV'$$
$$dV - \beta(1-p(i))dV' = u'(\tau - i)d\tau > 0$$

This tells us that, since marginal utility of the agent is positive, change in transfers goes in the same direction than change in level of utility to be provided. Transfers are decreasing for the successive trials in a given stage, since levels of utility to be provided are also decreasing. ■

Intuitively, we have that as V to be provided is smaller, V' becomes relatively cheaper and so the optimal contract provides relatively more promise than transfer for the new smaller level of utility to be provided. This makes difference between V and V' to decrease and so sequence of promised utilities to be a Cauchy sequence. For transfers, they decrease since they become relatively more expensive for the smaller utility to be provided in the successive trials.

4.2 Intermediate Stages:

At the intermediate stages, the recursive contract has three elements (τ, V'_{n+1}, V') : transfer, promised utility in case of success and promised utility in case of failure. The principal's problem has the form:

$$C_n^*(V) = \min_{\tau, V', V_{n+1}} \tau + \beta \left[p(i) C_{n+1}^*(V_{n+1}) + (1 - p(i)) C_n^*(V') \right]$$

$$s.t. V = u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right] \tag{9}$$

$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V') \tag{10}$$

$$V' \geqslant V_{\min}^n, V_{n+1} \geqslant V_{\min}^{n+1} \tag{11}$$

where $C_n^*(V)$ denotes the expected cost for the principal of the incentive compatible [10], promise keeping [9] and feasible [11] contract that provides the agent utility V.

Proposition 3 (Optimal contract intermediate stages) *Promised utilities in case of failure are decreasing with the level of utility to be provided for all $V > V_{\min}^n$. This sequence converges to $V_{\min}^n$. For $V = V_{\min}^n$ the optimal contract is stationary.*

Optimal contract offers decreasing sequences of transfers and promised utilities in case of success for the successive unsuccessful trials at a given intermediate stage as long as $V > V_{\min}^n$. For $V = V_{\min}^n$ the optimal contract is stationary.

Proof. Let $(V'(V), V_{n+1}(V), \tau(V))$ be the optimal contract that provides the agent utility V. We want to show that $V'(V) < V$ whenever $V > V_{\min}^n$.

In this case we have three choice variables, and to plug [9] and [10] (i.e. the expressions $i(V, V', V_{n+1})$ and $\tau(V, V', V_{n+1})$) into the principal's objective function allows us to choose two of them, $V'(V)$ and $V_{n+1}(V)$, and afterwards obtain $\tau(V)$ so that [9] is satisfied at V given $V'(V)$ and $V_{n+1}(V)$. Let $C(V)$ be the cost of this contract. We want to show that $V'(V) = V$ and $V'(V) > V$ lead to contradictions for any $V > V_{\min}^n$.

Step 1: $V'(V) = V$ is not optimal. Propose an alternative contract $\hat{\tau}(V) = \tau(V) + \delta$, $\hat{V}'(V) = V'(V) - \varepsilon$ and $\hat{V}_{n+1}(V) = V_{n+1}(V)$ that satisfies the promise keeping constraint at V. Change of agent's utility is given by:

$$dV = 0 = -\beta(1-p(i))\varepsilon + u'(\tau - i)\delta$$
$$\delta = \frac{\beta(1-p(i))}{u'(\tau - i)}\varepsilon$$

Let $\hat{C}(V)$ be the cost of this new contract. The difference in cost is given by

$$\begin{aligned}\hat{C}(V) - C(V) &= d\tau - \beta(1-p(i))\frac{dC_n(V')}{dV'}\varepsilon + \beta p'(i)\left(\frac{di}{d\tau}d\tau + \frac{di}{dV'}dV'\right)(C_{n+1}(V_{n+1}) - C_n(V')) = \\ &= -\beta p'(i)\frac{di}{dV'}\varepsilon(C_{n+1}(V_{n+1}) - C_n(V')) < 0\end{aligned}$$

The new contract is cheaper than the original one, what leads to a contradiction.

Step 2: $V'(V) > V$ is not optimal. Suppose not. In the same way we did in the previous step, we can propose an alternative contract with smaller promise and greater transfers that is cheaper. Cost function is increasing and convex, so in this case change in cost is greater than in the preceding step.

For $V = V^n_{\min}$, we have a stationary contract. To have $V'(V) < V$ is not feasible, and $V'(V) > V$ is not optimal, as shown in step 2.

Once $V'(V)$ is chosen, transfers and promised utilities in case of success are chosen so that [9] and first order conditions are satisfied. Cost function is increasing in each element of the contract, and first order conditions give the optimal choice among them. Optimally requires marginal cost of promises in case of success and failure to be equal, and is given by:

$$g(V', V_{n+1}) = \beta\left[(1-p(i))\frac{dC_{n+1}(V_{n+1})}{dV_{n+1}} - p(i)\frac{dC_n(V')}{dV'}\right] + \left[\frac{d\tau}{dV'} - \frac{d\tau}{dV_{n+1}}\right] = 0$$

Difference in marginal costs tomorrow has to adjust to compensate for the change in transfers needed to satisfy promise keeping constraint. From this optimally condition, we find that

$$\frac{dV'}{dV_{n+1}} > 0$$

We know that promise in case of failure is increasing with the utility level to be provided. From the first order conditions we learn that promise in case of success is also increasing with the level of utility to be provided. From [9] we can find optimal change in transfers for the successive trials given the optimal changes in promised utilities,

$$dV = u'(\tau - i)d\tau + \beta(1-p(i))dV' + \beta p(i)dV_{n+1}$$
$$dV - \beta(1-p(i))dV' - \beta p(i)dV_{n+1} = u'(\tau - i)d\tau > 0$$

From convexity of the cost functions, we learn that changes in promised utilities are decreasing with the level to be provided. Since marginal utility is positive, we find that transfers are increasing with the utility level to be provided. Since levels to be provided are decreasing with number of trials in a given stage, transfers are decreasing for the successive trials in a given stage. ■

4.3 First Stage:

At the first stage of the project, the principal chooses structure of the contract to be offered to the agent, that implicitly determined the level of utility provided to the agent when the contract is signed. We present the cost minimization problem of the principal for any level of utility to be provided, and once it is derived we choose the optimal level of utility the principal offers to the agent at the moment the contract is signed.

The principal's problem is to choose, for each level of utility V to be provided, $V \epsilon \left[V_{\min}^{1}, W\right]$ the contract that minimizes his cost, given by

$$
\begin{aligned}
C_1^*(V) = \min_{\tau, V', V_{n+1}} \; & \tau + \beta \left[p(i) C_2^*(V_2) + (1-p(i)) C_1^*(V')\right] \\
s.t. V = u(\tau - i) + \beta \left[p(i) V_2 + (1-p(i)) V'\right] & \quad (\lambda) \qquad (12) \\
u'(\tau - i) = \beta \left[p'(i)(V_2 - V')\right] & \quad (\mu) \qquad (13) \\
V' \geqslant V_{\min}^{1} & \quad (\gamma_1) \qquad (14) \\
V_2 \geqslant V_{\min}^{2} & \quad (\gamma_2) \qquad (15) \\
(\tau \geqslant \bar{\imath}) & \qquad (16)
\end{aligned}
$$

and the first order conditions of this problem are given by

$$
\begin{aligned}
FOC w.r.t.\tau &: \quad 1 - \lambda u'(\tau - i) + \mu u''(\tau - i) = 0 \\
FOC w.r.t.V' &: \quad \beta(1-p(i)) \frac{dC_2^*(V')}{dV'} + \beta p'(i)\mu - \beta(1-p(i))\lambda - \gamma_1 = 0 \\
FOC w.r.t.V_2 &: \quad \beta p(i) \frac{dC_2^*(V_2)}{dV_2} - \beta p'(i)\mu - \beta p(i)\lambda - \gamma_2 = 0 \\
Envelope &: \quad \frac{dC_1^*(V)}{dV} = \lambda \geqslant 0
\end{aligned}
$$

Rearranging these equations we get

$$
\begin{aligned}
\beta \left[\frac{dC_1^*(V')}{dV'} - \lambda\right] - \gamma_1 - \gamma_2 &= \beta p(i) \left[\frac{dC_1^*(V')}{dV'} - \frac{dC_2^*(V_2)}{dV_2}\right] \leqslant 0 \\
\frac{dC_1^*(V')}{dV'} &\leqslant \frac{dC_2^*(V_2)}{dV_2}
\end{aligned}
$$

When $\gamma_1 = 0$,

$$
\beta(1-p(i)) \left(\frac{dC_1^*(V')}{dV'} - \frac{dC_1^*(V)}{dV}\right) = -\beta p'(i)\mu \leqslant 0 \qquad (17)
$$

Since $C_1^*(V)$ is an increasing and convex function, $V_{\min}^{1} \leqslant V' \leqslant V$. When $\gamma_1 > 0$, from that trial on we have a stationary contract at $V = V' = V_{\min}^{1}$.

Claim 2 *Incentive compatibility constraint* [13] *binds and we have* $\mu > 0$ *whenever* $\gamma_1 = 0$

Proof. Suppose not, then we have

$$\frac{dC_1^*(V')}{dV'} = \frac{dC_1^*(V)}{dV}$$

but we can propose an alternative contract with $\hat{V}' = V' - \varepsilon_{V'}$ and $\hat{\tau} = \tau + \varepsilon_{\tau}$ that is cheaper, what leads to a contradiction. ■

Claim 3 $C_2^*(V_2) < C_1^*(V')$ *as long as constraint on promise in case of success in not binding.*

Proof. From first order conditions with respect to i we have that

$$\begin{aligned} \beta p'(i)\left[C_2^*(V_2) - C_1^*(V')\right] &= -\mu\left[-u''(\tau - i) - p''(i)(V_2 - V')\right] < 0 \\ C_2^*(V_2) &< C_1^*(V') \end{aligned}$$

■

Proposition 4 (Participation bonus first stage) *Promised utility to the agent at the starting point of the contract is given by* $V_{\min}^1$.

Proof. The principal chooses initial promise to the agent that is lest costly and that induces the agent to provide at least the minimum level of investment. Since cost of promised utility is an increasing function, the principal takes $V = V_{\min}^1$. Promised utility smaller than the initial level is not feasible, and greater would be more expensive. From [17], optimal contract to provide $V = V_{\min}^1$ is stationary. ■

Proposition 5 (Optimal transfers first stage) *Optimal contract at first stage is stationary: transfer and promised utility in case of success are constant for any number of trials until first stage is completed.*

Proof. From [17], optimal contract to provide $V = V_{\min}^1$ is stationary, and provides same transfer and promise in case of success for all trials until second stage is reached. ■

5 Extensions

We present in this section extensions to the model. The first extension introduces a exogenous positive probability of the project going backwards to a preceding stage. On the second extension we introduce a cost of state verification to be paid by the principal every period to obtain information about the situation of the project. And on the third extension we allow for intermediate stages to be valued by the principal.

5.1 Introduction of probability of falling to preceding stages.

We allow in this section for the possible destruction of stages already attained. The destruction of stages already completed is due to exogenous factors. Our objective is to see how the optimal contract adapts to this additional risk on the project.

Let α be the exogenous probability that the current stage of the project is destroyed, and let $\tilde{V}$ be the utility the contract provides to the agent at that situation.

The principal's problem at intermediate stage n becomes:

$$\begin{aligned} C_n^*(V) = \min_{\tau, V', V_{n+1}, \tilde{V}} \tau + \beta\left[p(i)C_{n+1}^*(V_{n+1}) + (1-p(i)-\alpha)C_n^*(V') + \alpha C_{n-1}^*(\tilde{V})\right] \\ s.t. V = u(\tau - i) + \beta\left[p(i)V_{n+1} + (1-p(i)-\alpha)V' + \alpha\tilde{V}\right] \\ u'(\tau - i) = \beta p'(i)(V_{n+1} - V\prime) \\ V' \geqslant V_{\min}^{n,\alpha}, V_{n+1} \geqslant V_{\min}^{n+1,\alpha} \end{aligned}$$

Where the vector $\{C^*\}$ is the fixed point of the operator

$$TC_n(V) = \min_{\tau, V', V_{n+1}, \tilde{V}} \tau + \beta\left[p(i)C_{n+1}(V_{n+1}) + (1-p(i)-\alpha)C_n(V') + \alpha C_{n-1}(\tilde{V})\right]$$

$$s.t. V = u(\tau - i) + \beta\left[p(i)V_{n+1} + (1-p(i)-\alpha)V' + \alpha\tilde{V}\right] \tag{18}$$

$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V\prime) \tag{19}$$

$$V' \geqslant V_{\min}^{n,\alpha}, V_{n+1} \geqslant V_{\min}^{n+1,\alpha} \tag{20}$$

and $V_{\min}^{s,\alpha}$ represents the minimum utility to be provided in a stationary contract at stage s so that agent's investment choice is at least the minimum one when probability of falling to previous stages is α. The vector $(C_1^*, C_2^*, C_3^{s+1}, ..., C_N^{s+1})$ is a vector of increasing, differentiable and convex functions. Proof is provided at Appendix B.

Claim 4 *Projects with greater probability of going backwards due to exogenous factors have greater transfers and promised utilities.*

Proof. The agent's investment decision [19] is not affected directly by the exogenous probability of going backwards. From promise keeping constraint [18], we find that

$$\frac{dV}{d\alpha} = \beta(\tilde{V} - V') < 0$$

To provide same utility level to the agent, transfers and promises for projects with greater α should be greater. Since both elements increase cost, principal chooses relative increases on each of them to satisfy the constraints with the cheapest available contract. ∎

5.2 Costly state verification

In this section we want to relax the assumption that the principal can verify for free the state the project is in after every trial. The consideration of a cost of state verification introduces an additional factor into the principal's problem: time. The longer it takes to finish the project, the greater is the burden of the cost of state verification.

We denote as c the cost the principal needs to pay each trial to verify the situation of the project. When this cost is introduced, the principal's problem at last stage becomes:

$$\begin{aligned} C^*_{Nc}(V) &= \min_{\tau, V'} (\tau + c) + \beta \left[(1 - p(i))C^*_{Nc}(V')\right] \\ s.t. V &= u(\tau - i) + \beta \left[p(i)W + (1 - p(i))V'\right] \\ u'(\tau - i) &= \beta \left[p'(i)(W - V')\right] \\ V' &\geqslant V^{N,c}_{\min} \end{aligned}$$

The first order conditions do not change, but the shape of the cost function does change.

Claim 5 *When there is a fixed cost that has to be paid in every transaction, the optimal contract prescribes greater transfers and smaller promised utilities.*

Proof. Denote by $(\tau_0(V), V'_0(V))$ and $(\tau_c(V), V'_c(V))$ the optimal contracts when there is no fixed cost and when there is a fixed cost c respectively. We proceed by contradiction:

Suppose $\tau_0(V) > \tau_c(V)$ and $V'_0(V) > V'_c(V)$ or $\tau_0(V) < \tau_c(V)$ and $V'_0(V) < V'_c(V)$. Both contracts should satisfy the same promise keeping constraint. But there is one situation (no cost or c cost respectively) where the promise keeping constraint is not binding. In that case, we can propose an alternative contract with smaller transfer and promised utility that is cheaper and satisfies the constraints, what leads to a contradiction.

Suppose $\tau_0(V) > \tau_c(V)$ and $V'_0(V) < V'_c(V)$. From the cost function, we know that when the cost is introduced, cost of promised utility becomes greater. We can propose an alternative contract with more transfer and smaller promised utility that is cheaper, and moreover induces more investment. That contradicts $(\tau_c(V), V'_c(V))$ to be optimal.

So, Optimal contract satisfies $\tau_0(V) < \tau_c(V)$ and $V'_0(V) > V'_c(V)$. ■

This result is intuitive: when the fixed cost is introduced, is good for the principal to try to incentive a greater investment so the project is successful sooner and the burden of the fixed cost decreases.

5.3 Upgrades: Agent valued intermediate stages

We relax in this section the assumption that the project is valued by the agent only when all stages have been completed. Let the agent value stage (N-1) by $W_{N-1} < W$, i.e. once stage (N-1) is completed, agent gets a flow utility of $(1 - \beta)W_{N-1}$ before last stage is completed. Once last stage is completed, flow utility from the whole project is given by $(1-\beta)W$. The principal's problem at stage (N-2) becomes:

$$C^*_{N-2}(V) = \min_{\tau,V',V_{N-1}} \tau + \beta\left[p(i)C^*_{N-1}(V_{N-1}) + (1-p(i))C^*_{N-2}(V')\right]$$

$$s.t. V = u(\tau - i) + \beta\left[p(i)V_N - 1 + (1-p(i))V'\right] \tag{21}$$

$$u'(\tau - i) = \beta\left[p'(i)(V_{N-1} - V')\right] \tag{22}$$

$$V' \geqslant V^{N-2}_{\min} \tag{23}$$

$$V_{N-1} \geqslant V^{N-1,W_{N-1}}_{\min} \tag{24}$$

and at last stage, principal's problem is:

$$\begin{gathered} C^*_{N-1}(V) = \min_{\tau,V'} \tau + \beta\left[(1-p(i))C^*_{N-1}(V')\right] \\ s.t. V = u(\tau - i) + W_{N-1}(1-\beta) + \beta\left[p(i)W + (1-p(i))V'\right] \\ u'(\tau - i) = \beta\left[p'(i)(W - V')\right] \\ V' \geqslant V^{N-1,W_{N-1}}_{\min} > W_{N-1} \\ (\tau \geqslant \bar{\imath}) \end{gathered}$$

We have that

$$\frac{dC^*_{N-1}(V)}{dW_{N-1}} < 0 \text{ as long as } V > V^{N-1,W_{N-1}}_{\min}$$

cost is decreasing with the agent's valuation of the intermediate stage.

Claim 6 *Transfers and promises in case of failure are smaller the more the intermediate stages are valued by the agent.*

Proof. Let $(\tau, \tilde{V}_{N-1}, \tilde{V}')$ be the optimal contract at stage (N-2) when $W_{N-1} = 0$ and utility to be provided is V.

If $\tilde{V}_{N-1} > V^{N-1,W_{N-1}}_{\min} > V^{N-1}_{\min,}$ then this contract is feasible when $W_{N-1} > 0$. But it is the case that promise in case of success becomes relatively cheaper when this stage is valued. We can propose an alternative contract $(\tau - \varepsilon_\tau, \tilde{V}_{N-1} + \varepsilon_{V_{N-1}}, V')$ that is cheaper. So, smaller transfers and promises in case of failure are provided.

If $V^{N-1,W_{N-1}}_{\min} > \tilde{V}_{N-1} > V^{N-1}_{\min,}$ the feasible contract when $W_{N-1} > 0$ needs $\hat{V}_{N-1} > V^{N-1,W_{N-1}}_{\min} > V^{N-1}_{\min,}$ and from [21] we find that $\hat{\tau} < \tau$. ■

6 Conclusions

The proposed contract offers an alternative to conditionality for situations where the principal cannot commit to abandoning the project before completion. In the case of foreign aid, conditionality clauses have not historically been honored. In the case of a division of a firm with important specific capital, the threat of firing a valuable division leader and abandoning the project is also an incredible threat. The proposed scheme does not present credibility problems (Samaritan's dilemma) for the principal, since it is assumed that the contract does not end until the project is completed.

On the other hand, the contract suits the situation where the agent values the completed project. In textbook moral hazard, the only instrument the principal has available to provide incentives to the agent are funds transfers. When the agent values the completed project, this value gives the agent incentive to invest, and that should be accounted for in the contract offered. The possibility of a bonus when a project is completed, or of a reward such as leadership of a better division, provides incentives to the agent to use the transferred funds appropriately. The value of the project aid finances for the government of a developing country gives the agent incentive to use the funds appropriately.

The extensions of the model make it flexible, allowing it to adapt to several frameworks. The introduction of the probability of falling introduces the idea of "risk sharing" in the model. The principal takes into account all possible situations that the dynamics of the project can take the agent to, and chooses promised utilities in each possible scenario. Greater transfers and promised utilities are offered for greater probabilities of going backwards in the project, which is reasonable given that the principal is aware that this is independent of the agent's investment decision.

The cost of checking the situation of the project highlights the fact that time is costly for the principal, and he adapts the contract accordingly, giving incentives to greater investment.

References

[1] **Alesina, Alberto and Dollar, David** *"Who gives aid to whom and why?"* Journal of economic growth 5 (2000).

[2] **Azam, Jean-Paul and Tirole, Jean-Jacques** *"Contracting for aid"* Journal of development economics 70 (2003)

[3] **Benveniste, L.M. and Scheinkman** *"On the differentiability of the value function in dynamic models of economics"* Econometrica 47 (1979)

[4] **Blackwell, David** *"Discounted dynamic programing"* Annals of Mathematical Statistics 36 (1965)

[5] **Buchanan, James M**. *"The Samaritans's dilemma".* In "Altruism, morality, and economic theory", editor Edmund S. Phelps. Russel Sage Foundation. (1975).

[6] **Burnside, Craig and Dollar, David** *"Aid, policies and growth"* The American Economic Review 90 (Sept. 2000)

[7] **Burnside, Craig and Dollar, David** *"Aid, policies and growth:Reply"* The American Economic Review 94 (June 2004)

[8] **Clemens M, Radelet S. and Bhavnani, R.** *"Counting chickens when they hatch: The short-term effect of aid on growth"* Working Paper 44. Center fro Global Development (2004).

[9] **Devarajan, Dollar, Holmgren** *"Aid and Reform in Africa"* The World Bank Publications 2001.

[10] **Dollar,David and Easterlly,Willian** *"In search of the key:Aid, Investment and Policies in Africa"* World Bank working paper (1999)

[11] **Drazen, Allan** *"What is gained by selectively withholding foreign aid?"* Unpublished manuscript. (April 1999)

[12] **Drehen, Alex** "The Development and Implementation of IMF and World Bank Conditionality." *Discussion Paper HWWA* (2002).

[13] **Easterly, William, Levine,Ross and Roodman, David** *"Aid, policies and growth:Comment"* The American Economic Review 94 (June 2004)

[14] **Easterly, William** *The Elusive Quest for Growth.* The MIT Press (2000).

[15] **Killick, Tony** "Aid and the Political Economy of Policy Change." With Gunatilaka, Ramani and Marr, Anna. Routledge (1998).

[16] **Spear, Stephen E. and Srivastava, Sanjay** "On repeated moral hazard with discounting." *Review of Economic Studies* 54 (4) 1987

[17] **Stockey, Nancy and Lucas,Robert** *Recursive methods in economic dynamics.* Harvard University Press.

[18] **Svenson, Jakob** *"When is foreign aid policy credible? Aid dependence and conditionality"* Journal of development economics 61 (2000)

[19] **Svenson, Jakob** *"Why conditional aid does not work and what can be done about it?"* Journal of development economics 70 (2003)

[20] **Werning, Ivan** "Repeated Moral Hazard with Unmonitored Wealth: A Recursive First Order Approach." (2000) Unpublished manuscript.

[21] **Werning, Ivan** "Optimal Unemployment Insurance with unobservable savings." (2002) Unpublished manuscript.

[22] **The World Bank** Assessing Aid: What works, what doesn't and why. November 1998.

[23] **The World Bank** Review of World Bank Conditionality. July 2005.

[24] **The World Bank** Conditionality Revised: Concepts, Experiences and Lessons (2005)

A Appendix

Let $X \subset \mathbb{R}$ be a compact set. T is an operator on $\mathcal{C}(X)$, the space of bounded and continuous functions $f : X \to \mathbb{R}$.

$$T : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$$

$$\begin{gathered}
TC_n(V) = \min_{\tau, V', V_{n+1}} \tau + \beta \left[p(i) C^*_{n+1}(V_{n+1}) + (1 - p(i)) C_n(V') \right] \\
s.t. V = u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right] \\
u'(\tau - i) = \beta p'(i)(V_{n+1} - V') \\
V' \geqslant V^n_{\min}, V_{n+1} \geqslant V^{n+1}_{\min}
\end{gathered}$$

Proposition 6 *T operator has a fixed point. The fixed point is a vector of continuous functions.*

Proof. We have a T operator for each stage. Since $C_n(x)$ appears on $C_{n-1}(x)$, we solve the problem backwards. We start at last stage, and we use the properties of $C_N(x)$ backwards in the successive $C_n(x)$ for all $n < N$. We show here the argument for a generic stage $n < N$.

To show that the T operator has a fixed point, we need to show that it is a contraction on a complete metric space. Since $\mathcal{C}(X)$ is a complete metric space, it suffices to show that Blackwell (65) sufficient conditions for a contraction are satisfied.

Monotonicity: For any pair of functions $f, g \,\epsilon\, \mathcal{C}(X)$, such that $f(x) \leqslant g(x)$ for all $x \in X$ we have $Tf(x) \leqslant Tg(x)$ for all $x \in X$.

$$\begin{aligned}
Tf_k(v) &= \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) f_{k+1}(V_{k+1}) + (1 - p(i)) f_k(V') \right] \\
&\leqslant \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) g_{k+1}(V_{k+1}) + (1 - p(i)) g_k(V') \right] = Tg_k(v)
\end{aligned}$$

Discounting: Exists a $\beta \in (0, 1)$ s.t. $T(f + a)(x) \leqslant Tf(x) + \beta a$ for all $f \in C^N$, $a \geqslant 0, x \in X$.

$$\begin{aligned}
T(f_k + a)(v) &= \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i)(f_{k+1}(V_{k+1}) + a) + (1 - p(i))(f_k(V') + a) \right] \\
&\leqslant \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) f_{k+1}(V_{k+1}) + (1 - p(i)) f_k(V') \right] + \beta a = T(f_k)(v) + \beta a
\end{aligned}$$

Our mapping is a contraction in a complete metric space, so it has a fixed point. The fixed point is a continuous function since it belongs to $\mathcal{C}(X)$. ■

Proposition 7 *The fix point is an increasing, differentiable and convex function.*

Proof. Step1: The fixed point is an increasing function. Suppose not. Take $V_1 = V_0 - \varepsilon$. Decrease V' and τ so that incentive compatibility constraint is satisfied (investment does not change),

$$\begin{aligned}
u'(\tau - \Delta\tau - i) &= \beta \left[p'(i)(W - (V\prime - \Delta V')) \right] \\
-u''(\tau - i)\Delta\tau &= \beta p'(i)\Delta V' \\
\Delta V' &= -\frac{u''(\tau - i)\Delta\tau}{\beta p'(i)}
\end{aligned}$$

We need to check that promise keeping constraint is satisfied:

$$\begin{aligned}\Delta V &= -u'(\tau-i)\Delta\tau - \beta(1-p(i))\Delta V' \\ \Delta V &= \Delta\tau\left[-u'(\tau-i) + \frac{(1-p(i))u''(\tau-i)}{p'(i)}\right] = -\varepsilon < 0\end{aligned}$$

What implies that, starting with a $C(.)$ increasing, we have

$$T_n(C(V_1)) > T_n(C(V_0))$$

so the operator maps increasing functions into increasing functions.

Step 2: The fixed point is a convex function.

To show differentiability we apply Benveniste Sheinkman (79) theorem. Define the function $W_n : X \to \mathbb{R}$ as

$$\begin{aligned}W_n(V) &= \tilde{\tau}(V, V'(V_0), V_{n+1}(V_0)) + \\ &+\beta p(i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0)))C_{n+1}(V_{n+1}(V_0)) + \\ &+\beta(1 - p(i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0)))C_n(V'(V_0))\end{aligned}$$

where $\tilde{\tau}(V, V'(V_0), V_{n+1}(V_0))$ denotes the transfer needed to provide a lifetime utility V given $V'(V_0)$ and $V_{n+1}(V_0)$. Investment that this transfer and the given $V'(V_0)$ and $V_{n+1}(V_0)$ induces is denoted by $i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0))$.

$W_n(V)$ is a convex function given the assumptions on utility functions and probabilities.

Let D be a neighborhood of V_0. $W_n(V_0) = C_n(V_0)$, and $W_n(V) \geqslant C_n(V)$ for all $V \in D$. Then,by Benveniste Sheinkman (79), C_n is differentiable at V_0 and $C_n'(V_0) = W_n'(V_0)$

$$\begin{aligned}W_n'(V_0) &= \frac{d\tilde{\tau}(V, V'(V_0), V_{n+1}(V_0))}{dv} \\ &+\beta p'(i)\frac{di(\tilde{\tau}, V'(V_0), V_{n+1}(V_0))}{dv}(C_{n+1}(V_{n+1}(V_0)) - C_n(V'(V_0))) \\ W_n''(V_0) &= C_n''(V_0) > 0\end{aligned}$$

So $C_n(V)$ is a differentiable, increasing and convex function. ■

Where the vector $\{C^*\}$ is the fixed point of the operator

$$TC_n(V) = \min_{\tau, V', V_{n+1}, \tilde{V}} \tau + \beta\left[p(i)C_{n+1}(V_{n+1}) + (1-p(i)-\alpha)C_n(V') + \alpha C_{n-1}(\tilde{V})\right]$$

$$s.t. V = u(\tau - i) + \beta\left[p(i)V_{n+1} + (1-p(i)-\alpha)V' + \alpha\tilde{V}\right]$$

$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V\prime)$$

$$V' \geqslant V_{\min}^{n,\alpha}, V_{n+1} \geqslant V_{\min}^{n+1,\alpha}$$

and $V_{\min}^{s,\alpha}$ represents the minimum utility to be provided in a stationary contract at stage s so that agent's investment choice is at least the minimum one.

B Appendix

Let $X \subset \mathbb{R}^N$ be a compact set. T is an operator on $\mathcal{C}^N(X)$, the space of vectors of N bounded and continuous functions $f : X \to \mathbb{R}^N$.

$$T : \mathcal{C}^N(X) \longrightarrow \mathcal{C}^N(X)$$
$$T(C_1^s, C_2^s, C_3^s, ..., C_N^s) = (T_1(C_1^s, C_2^s, C_3^s, ..., C_N^s), , ..., T_N(C_1^s, C_2^s, C_3^s, ..., C_N^s)) =$$
$$= (C_1^{s+1}, C_2^{s+1}, C_3^{s+1}, ..., C_N^{s+1})$$

$$TC_n(V) = \min_{\tau, V', V_{n+1}, \tilde{V}} \tau + \beta \left[p(i)C_{n+1}(V_{n+1}) + (1 - p(i) - \alpha)C_n(V') + \alpha C_{n-1}(\tilde{V}) \right]$$
$$s.t. V = u(\tau - i) + \beta \left[p(i)V_{n+1} + (1 - p(i) - \alpha)V' + \alpha \tilde{V} \right]$$
$$u'(\tau - i) = \beta p'(i)(V_{n+1} - V\prime)$$
$$V' \geqslant V_{\min}^{n,\alpha}, V_{n+1} \geqslant V_{\min}^{n+1,\alpha}$$

Proposition 8 *T operator has a fixed point. The fixed point is a vector of continuous functions.*

Proof. To show that this operator has a fixed point, we need to show that it is a contraction in a complete metric space. We extend Stokey and Lucas proof to a vector of functions as follows:

Step 1: $\mathcal{C}^N$ is complete under the following norm. For $F, G \in \mathcal{C}^N$,

$$\|F(x) - G(x)\| = \max_j \sup_{x \epsilon X} |f_j(x) - g_j(x)|$$

Take a Cauchy sequence of functions $\{F^k\}$ .We need to show that this sequence converges to $F \in \mathcal{C}^N$, i.e., there exists $F \in \mathcal{C}^N$ such that for any $\varepsilon > 0$ there exists N_ε such that $\|F^n - F\| \leq \varepsilon$, all $n \geqslant N_\varepsilon$.

We proceed as follows: Using completeness of $\mathbb{R}_0^N$, we find a candidate for convergence point. We show that $\{F^k(x)\}$ converges to the candidate point, and we complete the proof showing that the candidate point is indeed a vector of continuous functions.

Fix $x \in X$. $\{F^k(x)\}$defines a sequence in $\mathbb{R}^N$. Since $\mathbb{R}_0^N$ is a complete metric space[6], the Cauchy sequences for each $x \in X$ converge to $F(x) \in \mathbb{R}^N$. Take this $F(x)$ as a candidate for convergence point.

To show that $\{F^k\}$ converges to F, we need to show that as $k \to \infty$, $\left\|F^k(x) - F(x)\right\| \to 0$.

For a fixed $x \in X$,choose N_ε so that $n, m > N_\varepsilon$ implies $\|F^n(x) - F^m(x)\| \leqslant \frac{\varepsilon}{2}$.

$$\|F^n(x) - F(x)\| \leqslant \|F^n(x) - F^m(x)\| + \|F^m(x) - F(x)\| \leqslant \varepsilon \text{ for all } n \geqslant N_\varepsilon$$

and this holds for all $x \in X$.

What is left to show is that $F(x)$ is a continuous function.

[6] $\mathbb{R}^N$ with the norm $\|x - y\| = \max_{1 \leqslant k \leqslant N} |x_k - y_k|$

Take ε and x given. Choose Ψ so that$\left\| F - F^{\Psi} \right\| < \frac{\varepsilon}{3}$ (since the sequence $\left\{ F^{k}(x) \right\}$ converges to F(x), such a Ψ exists).

Choose δ so that $\|x-y\| < \delta$ implies $\left\| F^{\Psi}(x) - F^{\Psi}(y) \right\| < \frac{\varepsilon}{3}$ (possible since F_{Ψ} is a vector of continuous functions).

$$\begin{aligned} \|F(x) - F(y)\| &\leqslant \left\| F(x) - F^{\Psi}(x) \right\| + \left\| F^{\Psi}(x) - F^{\Psi}(y) \right\| + \left\| F^{\Psi}(y) - F(y) \right\| \\ &\leqslant 2 \left\| F - F^{\Psi} \right\| + \left\| F^{\Psi}(x) - F^{\Psi}(y) \right\| < \varepsilon \end{aligned}$$

Since F is a continuous function, we conclude that $\mathcal{C}^N$ is complete.

Step 2: Blackwell (65) sufficient conditions for a contraction.

Monotonicity: For any pair of functions $f, g \ \epsilon \ \mathcal{C}^N$, such that $f_j(x) \leqslant g_j(x)$ for all $x \in X$, $j : 1, ..., N$ we have $Tf(x) \leqslant Tg(x)$ for all $x \in X$, $j : 1, ..., N$.

$$\begin{aligned} Tf_k^j &= \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) f_{k+1}^j(V_{k+1}) + (1 - p(i) - \alpha) f_k^j(V') + \alpha f_k^j(\tilde{V}) \right] \\ &\leqslant \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) g_{k+1}^j(V_{k+1}) + (1 - p(i) - \alpha) g_k^j(V') + \alpha g_k^j(\tilde{V}) \right] = Tg_k^j \end{aligned}$$

Discounting: Exists a $\beta \in (0,1)$ s.t. $T(f+a)(x) \leqslant Tf(x) + \beta a$ for all $f \in \mathcal{C}^N$, $a \geqslant 0, x \in X$.

$$\begin{aligned} T(f_k^j + a)(v) &= \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i)(f_{k+1}^j(V_{k+1}) + a) + (1 - p(i) - \alpha)(f_k^j(V') + a) + \alpha (f_k^j(\tilde{V}) + a) \right] \\ &\leqslant \min_{\tau, V', V_{k+1}} \tau + \beta \left[p(i) f_{k+1}^j(V_{k+1}) + (1 - p(i) - \alpha) f_k^j(V') + \alpha f_k^j(\tilde{V}) \right] + \beta a = T(f_k^j)(v) + \beta a \end{aligned}$$

Our mapping is a contraction in a complete metric space, so it has a fixed point. The fixed point is a continuous function. ■

Proposition 9 *The fix point is a vector of increasing, differentiable and convex functions.*

Proof. Step 1: The fixed point is a vector of increasing functions.

Take the $n^{th.}$ function at iteration j. $(C_1^j, C_2^j, C_3^j, ..., C_N^j)$ is a vector of increasing functions. Show that $C_n^{j+1}$is also an increasing function for all n.

$$\begin{gathered} C_n^{j+1}(V_1) = T_n(C_1^j, C_2^j, C_3^j, ..., C_N^j)(V_1) = \\ = \min_{\tau, V', V_{n+1}} \tau + \beta \left[p(i) C_{n+1}^j(V_{n+1}) + (1 - p(i) - \alpha) C_n^j(V') + \alpha C_{n-1}^j(\bar{V}) \right] \\ s.t. V = u(\tau - i) + \beta \left[p(i) V_{n+1} + (1 - p(i)) V' \right] \\ u'(\tau - i) = \beta \left[p'(i)(V_{n+1} - V\prime) \right] \end{gathered}$$

Take $V_1 = V_0 - \varepsilon$. Decrease $V\prime$ and τ so that IC is satisfied (investment does not change),

$$\begin{aligned} u'(\tau - \Delta\tau - i) &= \beta \left[p'(i)(W - (V\prime - \Delta V')) \right] \\ -u''(\tau - i)\Delta\tau &= \beta p'(i) \Delta V' \\ \Delta V' &= -\frac{u''(\tau - i)\Delta\tau}{\beta p'(i)} \end{aligned}$$

We need to check that PK is satisfied:

$$\begin{aligned}
\Delta V &= -u'(\tau - i)\Delta\tau - \beta(1 - p(i) - \alpha)\Delta V' \\
\Delta V &= \Delta\tau\left[-u'(\tau - i) + \frac{(1 - p(i) - \alpha)u''(\tau - i)}{p'(i)}\right] = -\varepsilon < 0
\end{aligned}$$

What implies that, starting with a C(.) increasing, we have

$$T_n(C_1^j, C_2^j, C_3^j, ..., C_N^j)(V_1) > T_n(C_1^j, C_2^j, C_3^j, ..., C_N^j)(V_0)$$

so the operator maps increasing functions into increasing functions.

Step 2: The fixed point is a vector of convex functions.

To show differentiability we apply Benveniste Sheinkman theorem to each of the functions in the vector.

Let $\tilde{X}_n$ be the projection of X in its n^{th} component.

Take the function $W_n : \tilde{X}_n \to \mathbb{R}$ defined as

$$\begin{aligned}
W_n(V) &= \tilde{\tau}(V, V'(V_0), V_{n+1}(V_0)) + \\
&\quad + \beta p(i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0)))C_{n+1}(V_{n+1}(V_0)) + \\
&\quad + \beta(1 - p(i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0)) - \alpha)C_n(V'(V_0)) + \\
&\quad + \beta\alpha C_{n-1}(\bar{V})
\end{aligned}$$

where $\tilde{\tau}(V, V'(V_0), V_{n+1}(V_0))$ denotes the transfer needed to satisfy Promise Keeping V given $V'(V_0)$ and $V_{n+1}(V_0)$.Investment that this transfer together with the given $V'(V_0)$ and $V_{n+1}(V_0)$ induces is denoted by $i(\tilde{\tau}, V'(V_0), V_{n+1}(V_0))$.

$W_n(V)$ is a convex function given the assumptions on utility functions and probabilities.

Let D be a neighborhood of V_0. $W_n(V_0) = C_n(V_0)$, and $W_n(V) \geqslant C_n(V)$ for all $V \in D$.

Then,by Benveniste Sheinkman (79), C_n is differentiable at V_0 and $C_n'(V_0) = W_n'(V_0)$

$$\begin{aligned}
W_n'(V_0) &= \frac{d\tilde{\tau}(V, V'(V_0), V_{n+1}(V_0))}{dv} \\
&\quad + \beta p'(i)\frac{di(\tilde{\tau}, V'(V_0), V_{n+1}(V_0))}{dv}(C_{n+1}(V_{n+1}(V_0)) - C_n(V'(V_0))) \\
W_n''(V_0) &= C_n''(V_0) > 0
\end{aligned}$$

So $C_n(V)$ is a differentiable, increasing and convex function. ■