Auctions with Almost Homogeneous Bidders

by

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Abstract

We deviate from the symmetric case of the independent private value model by allowing the bidders' value distributions, which depend on parameters, to be slightly different. We show that previous results about the equality to the first-order in the parameters between revenues from the second-price auction and other auction mechanisms follow from the joint differentiability of the equilibria with respect to the parameters. We prove this differentiability for the first-price auction and obtain general formulas for the different first-order effects. From our results about the first-price auction, we analytically generate examples with continuous distributions where a stochastic improvement to a bidder's value distribution reduces his equilibrium payoff. In another application, we show that, starting from competition among cartels of equal sizes, allowing in a small number of members from other cartels can be profitable only if the members or the synergies between them are strong enough.

1. Introduction

Starting from the standard independent private value model with homogeneous bidders, Fibich, Gavious, and Sela (2004) consider a particular asymmetric perturbation of the valuation distributions. They show heuristically that the revenues from the second-price auction and from some other auctions, if equal with homogeneous bidders, are equal to the first-order in the size of their asymmetric perturbation. By increasing the dimensionality of the parameters that determine the perturbations, we show that Fibich *et al.* (2004)'s result is an immediate consequence of the joint differentiability with respect to these asymmetry parameters. We then go on to prove this differentiability for the first-price auction and for general asymmetric perturbations of the valuation distributions[1]. Formulas for the first-order effects on all equilibrium functions and quantities of interest can then be easily derived and extend the expressions Fibich and Gavious (2003) compute.

We next show two applications, pertaining to the first-price auction, of our results. In the first application, we analitically generate, from our explicit formulas for the first-order effects, a class of examples where the stronger a bidder is, the smaller his exante payoff becomes. Arozamena and Cantillon (2004) check numerically that an example they construct satisfies this property. The existence of continuous examples with this property already followed from the discrete examples Thomas (1997) found analytically and the continuity of the equilibrium with respect to the value distributions (see Lebrun 2002).

In the second application, we consider the formation of coalitions. We show that, starting from competing homogeneous cartels, allowing a small number of transfers from other cartels is profitable to a cartel only if the bidders or the synergies between them are strong enough. A result from

[1] Contrary to what Fibich *et al.* (2004) write in their footnote 4, Lebrun (1996, 1999) does not prove the differentiability with respect to the parameters.

Waehrer (1999) that compares the players' expected payoffs within the same equilibrium, combined with our differentiability result, implies that letting in a small number of weak bidders cannot increase the average payoff in the absence of significant synergies. Our explicit formulas show that the average payoff actually decreases in this case.

2. First-Order Revenue Equivalence

Consider the standard independent private value model with n risk-neutral bidders whose values are distributed over the same interval $[c, d]$. The value distributions $F_1, ..., F_n$ depend on nm parameters $\tau_1^1, ..., \tau_m^1; ...; \tau_1^n, ..., \tau_m^n$ in $(-\rho, \rho)$ through the function $F(.;.)$ as follows:

$$F_i(.) = F\left(.; \tau_1^i, ..., \tau_m^i\right), (1)$$

where m is a strictly positive integer, $\rho > 0$, and $(\tau_1^i, ..., \tau_m^i)$ is the vector τ^i of parameters specific to bidder i, for all $1 \leq i \leq n$. Throughout the paper, $F(.; \tau_1^i, ..., \tau_m^i)$ is a continuous-from-the-right cumulative distribution function with support $[c, d]$ and such that its restriction to $[c, d]$ is absolutely continuous, for all values of the parameters $\tau_1^i, ..., \tau_m^i$.[2]

Let $R^S(\tau^1, ..., \tau^n)$ be the auctioneer's expected revenues from the equilibrium in weakly dominant strategy–the sincere-bidding equilibrium–of the second-price auction when bidder i's vector of parameters is τ^i, for all i. In this case, we also denote $R^M(\tau^1, ..., \tau^n)$ the expected revenues from an incentive compatible and individually rational direct mechanism M. The following proposition is an simple mathematical exercise. In its statement as in the rest of the paper, a bold character denotes a nm-dimensional vector.

[2] While in the main text we will assume $F\left(.; \tau_1^i, ..., \tau_m^i\right)$ to be atomless, in the appendices we will sometimes allow a mass point at c.

Proposition 1: *Assume R^M is symmetric, that is, $R^M(\tau^1, ..., \tau^n) = R\left(\tau^{\pi(1)}, ..., \tau^{\pi(n)}\right)$, for all permutation π of $\{1, 2, ..., n\}$. Assume further that the revenues R^M and R^S coincide when the values are distributed identically, that is, $R^M(\tau, ..., \tau) = R^S(\tau, ..., \tau)$, for all τ in $(-\rho, \rho)^m$. Then, if R^M and R^S are differentiable at $\mathbf{0}$, where $\mathbf{0}$ is the null nm-dimensional vector, whose all components are equal to 0, we have (i) and (ii) below:*

(i) The difference between $R^M(\boldsymbol{\tau})$ and $R^S(\mathbf{0})$ is of the first-order in the parameters, that is:

$$R^M(\boldsymbol{\tau}) = R^S(\mathbf{0}) + o(|\boldsymbol{\tau}|);$$

(ii) For $\boldsymbol{\delta} = \left(\delta^1, ..., \delta^n\right) \in R^{nm}$ different from the null vector, the derivative of R^M at $\mathbf{0}$ in the direction $\boldsymbol{\delta}$ is equal to the derivative of R^S at $\mathbf{0}$ in the direction $\left(\sum_{i=1}^n \delta^i/n, ..., \sum_{i=1}^n \delta^i/n\right)$.

Proof: See Appendix 1.

If the direct mechanism M is constructed from equilibrium strategies of some auction procedure, the revenue function R^M is symmetric when the auction's rules are anonymous and the equilibrium unique for all n-tuples of distributions $(F(.;\tau^1), ..., F(.;\tau^n))$ with $(\tau^1, ..., \tau^n)$ in $(-\rho, \rho)^{nm}$. From the Revenue Equivalence Theorem (see Myerson, 1981), the functions R^M and R^S coincide for identical value distributions if, in those cases, M allocates the item to the highest value bidder and leaves no payoff to any bidder with value c.[3]

To illustrate Proposition 1 and its proof, consider the case with two bidders and one-dimensional parameters, that is, $n = 2$ and $m = 1$. By assumption, the mechanism M and the second-price auction give the same revenues when both bidders' values are distributed according to $F(.;\tau)$, that

[3]For example, in Fibich *et al* (2004)'s auctions: anonimity comes from the explicit assumption that the winner be the highest bidder; efficient allocation in the symmetric case and uniqueness are implictly assumed; and zero payoff to the bidder with the lowest possible value is explictly assumed.

is, when the couple of parameters (τ^1, τ^2) is equal to (τ, τ) and hence belongs to the diagonal D (see Figure 1). Consequently, the derivatives of R^M and R^S at $(0,0)$ in the direction $(1,1)$ along this diagonal are identical.

FIGURE 1

The derivative of R^S at $(0,0)$ in the direction $(1,-1)$ is equal to zero. If it was strictly positive, for example, there would exist a small $\lambda > 0$ such that $R^S(\lambda, -\lambda)$ would be strictly larger than $R^S(-\lambda, \lambda)$. However, this is impossible since the revenues from the second-price auction when the couple of value distributions is $(F(.;\lambda), F(.;-\lambda))$ are the same as when the distribution couple is $(F(.;-\lambda), F(.;\lambda))$, for all λ. Only the labeling of the bidders differs, with no effect on the total revenues.

The assumed symmetry of R^M implies similarly that the derivative of R^M in the direction $(1,-1)$ is equal to zero. Consequently, all directional derivatives for R^M and R^S are identical. Moreover, the difference between $R^M(\tau^1, \tau^2)$ and $R^S(\tau^1, \tau^2)$ vanishes at $(\tau^1, \tau^2) = (0,0)$. Thus, from the assumption of differentiability, this difference is equal to zero to the first-order in the distance between (τ^1, τ^2) and $(0,0)$, that is, the ratio $\frac{\left|R^M(\tau^1,\tau^2) - R^S(\tau^1,\tau^2)\right|}{|(\tau^1,\tau^2)|}$ tends towards zero as the length $|(\tau^1, \tau^2)|$ of the vector (τ^1, τ^2) tends towards zero. Proposition 1 (i) follows.

Since the derivative of R^M in the direction orthogonal to D vanishes, its derivative along any direction (δ^1, δ^2) is equal to its derivative along the orthogonal projection $\frac{\delta^1+\delta^2}{2}(1,1)$ onto D. Proposition 1 (ii) follows and

$$R^M\left(\varepsilon\delta^1, \varepsilon\delta^2\right) = R^S(0,0) + \varepsilon\frac{\delta_1+\delta_2}{2}\left(\frac{d}{d\tau}R^S(\tau,\tau)\right)_{\tau=0} + o(\varepsilon).$$

In the next section, we show conditions under which Proposition 1 may be applied to the case where M is the equilibrium of the first-price auction and prove that the examples below can be made to satisfy these conditions.

Example 1: Consider Fibich *et al* (2004)'s asymmetric perturbations:

$$F_i(.) = F(.) + \varepsilon H_i(.), \ (2)$$

where F is an absolutely continuous cumulative distribution function[4]. Their results follow from Proposition 1 if we consider rather the following multidimensional perturbations:

$$F\left(.;\tau_1^i,...,\tau_n^i\right) = F(.) + \sum_{k=1}^{n} \tau_k^i H_k(.), \ (3)$$

with $(\tau_1^i,...,\tau_n^i) \in (-\rho,\rho)^n$, H_k is continuous and such that $H_k(c) = H_k(d) = 0$, for all $1 \leq k \leq n$, and $\frac{d}{dv}F - \rho \sum_{k=1}^{m} \left|\frac{d}{dv}H_k\right|$ exists and is strictly positive over $(c,d]$. The departure (2) from the symmetric model is then the particular case of (3) where $(\tau_1^i,...,\tau_n^i) = \varepsilon\left(\delta_1^i,...,\delta_n^i\right)$, with $\delta_j^i = 0$ if $j \neq i$ and $\delta_i^i = 1$ (here, in the notation of Proposition 1, $m = n$).

From Proposition 1 (ii), assuming differentiability, the derivative of R^M at the origin $\mathbf{0}$ of the parameter space in the direction $\left(\delta^1,...,\delta^n\right)$ is the derivative of R^S in the direction $\left(\sum_{i=1}^n \delta^i/n,...,\sum_{i=1}^n \delta^i/n\right) = (e,...,e)/n$, where e is the n-dimensional vector with all its components equal to 1.

When the vector of parameters is $\varepsilon/n\,(e,...,e)$, every distribution function F_i in (3) is $G(.;\varepsilon) = F(.) + \varepsilon \sum_{k=1}^n H_k(.)/n$, whose derivative with respect to ε is equal to $\sum_{k=1}^n H_k(.)/n$. From the formula $d - \int_c^d \left\{nG(.;\varepsilon)^{n-1} - (n-1)\,G(.;\varepsilon)^n\right\} dv$ for the revenues from the second-price auction in the symmetric case, the derivative with respect to ε at $\varepsilon = 0$ gives[5] the following value for the direc-

[4] Fibich *et al* (2004) actually assume that F is continuously differentiable, and that $|H_i| \leq 1$ and $H_i(c) = H_i(d) = 0$, for all $1 \leq i \leq n$. Obviously, we must, as we do below, add conditions to make sure that $F_1,...,F_n$ are probability distributions.

[5] In Section 3, we return to Example 1 and show (by appealing to Proposition 2, Section 3) that we may differentiate under the integral sign.

tional derivative of R^M:

$$-(n-1)\int_c^d F(v)^{n-2}(1-F(v))\left(\sum_{k=1}^n H_k(v)\right)dv,$$

the first-order effect found by Fibich *et al* (2004).

Example 2: In Section 4 especially, we will use the following departures from the symmetric setting with absolutely continuous distribution function F:

$$F\left(.;\tau_1^i,...,\tau_m^i\right) = F(.) . \prod_{k=1}^m H_k(.)^{\tau_k^i}, \quad (4)$$

where $(\tau_1^i,...,\tau_m^i) \in (-\rho,\rho)^m$, H_k is strictly positive and bounded over $(c,d]$, and such that $H_k(d) = 1$, for all $1 \leq k \leq m$, and $\frac{d}{dv}\ln F - \rho\sum_{i=1}^m \left|\frac{d}{dv}\ln H_i\right|$ exists and is strictly positive over $(c,d]$. When $m = n$, the derivative of the total revenues in the same direction $\left(\delta^1,...,\delta^n\right)$ as in Example 1 above, which gives the first-order effect of the deviation $F_i = FH_i^{\varepsilon}$, is[6]:

$$-(n-1)\int_c^d F(v)^{n-1}(1-F(v))\left(\sum_{k=1}^n \ln H_k(v)\right)dv.$$

3. The Second-Price and the First-Price Auctions

From the simplicity of the sincere-bidding equilibrium of the second-price auction, conditions under which R^S is differentiable are easily obtained. We have Proposition 2 below.

Proposition 2: *Assume there exists $0 < \rho' < \rho$ such that the distribution function (1) is absolutely continuous in v everywhere and continuously*

[6] In Section 3, we also return to Example 2 and prove (from Proposition 2, Section 3) that differentiation may be taken under the integral sign.

differentiable with respect to τ *in* $(-\rho', \rho')^m$ *and its partial derivatives with respect to* $\tau_1, ..., \tau_m$ *are bounded over* $(c, d) \times (-\rho', \rho')^m$*. Then,* R^S *is continuously differentiable over* $(-\rho', \rho')^m$ *and differentiation may be taken under the integral signs in the equality below (obtained through integration by parts):*

$$R^S\left(\tau^1, ..., \tau^n\right) = d + (n-1)\int_c^d \prod_{j=1}^n F\left(v; \tau^j\right) dv - \sum_{h=1}^n \int_c^d \prod_{\substack{j=1 \\ j \neq h}}^n F\left(v; \tau^j\right) dv.$$

Proof: For all $1 \leq i \leq n$ and $1 \leq k \leq m$, since $\frac{\partial}{\partial \tau_k} F(v; \tau)$ and $F(v; \tau)$ are bounded, when we differentiate the equality above with respect to τ_k^i, we may differentiate under the integral signs (for example, from Lebesgue theorem of dominated convergence), and we find:

$$\begin{aligned} \frac{\partial}{\partial \tau_k^i} R^S\left(\tau^1, ..., \tau^n\right) &= (n-1)\int_c^d \left(\frac{\partial}{\partial \tau_k} F\left(v; \tau^i\right)\right) \prod_{\substack{j=1 \\ j \neq i}}^n F\left(v; \tau^j\right) dv \\ &\quad - \sum_{\substack{h=1 \\ h \neq i}}^n \int_c^d \left(\frac{\partial}{\partial \tau_k} F\left(v; \tau^i\right)\right) \prod_{\substack{j=1 \\ j \neq h,i}}^n F\left(v; \tau^j\right) dv. \end{aligned}$$

Again because $\frac{\partial}{\partial \tau_k} F(v; \tau)$ and $F(v; \tau)$ are bounded, the equality above implies that $\frac{\partial}{\partial \tau_k^i} R^S(\tau^1, ..., \tau^n)$ is continuous. Proposition 2 follows. ||

From the previous section, if the equilibrium of the first-price auction is unique and differentiable with respect to the perturbation parameters, the equality to the first-order between the revenues from the first and second-price auctions immediately follows from the Revenue-Equivalence Theorem. From Lebrun (1999), under Assumption E below, any equilibrium is pure and satisfies a system of differential equations, obtained from the first-order conditions, with partially determined boundary conditions. From Lebrun (2006), Assumption U is an example of assumptions under which the equilibrium is unique. It requires that the value distributions' inverse hazard rates be strictly decreasing over an interval, however small, in the bottom of the valuation interval[7]. In Appendix 3, we prove many of our results under more general assumptions that allow a mass point at c, as in the presence of a binding reserve price (in which case, we may assume that the probability spread below the reserve price is rather concentrated at it.)

Assumption E: $F(.;\tau)$ is atomless and is differentiable–with respect to v–over $(c,d]$ and its derivative[8]–the density function $f(.;\tau)$–is locally bounded away from zero over this interval, for all τ in $(-\rho,\rho)^m$.

Assumption U: For all τ in $(-\rho,\rho)^m$, there exists $\pi > 0$, which may depend on τ, such that $F(.;\tau)$ is strictly log-concave over $(c, c+\pi)$.

As we show below, the differentiability of the equilibrium of the first-price auction and of R^F with respect to the perturbation parameters follows from Assumption D below. Since Assumption D immediately implies the assumptions of Proposition 2, R^S too is differentiable under D and the first-order equality between R^S and R^F follows from Proposition 1 (Section 2).

Assumption D:

(i) $F(v;\tau)$ can be extended beyond $v = d$ such that it is continuously differentiable–with respect to $(v;\tau)$–and $\frac{\partial}{\partial v}F(v;\tau)$ is strictly positive over an open set $(c, d+\zeta) \times (-\rho,\rho)^m$, where $\zeta > 0$.

(ii) There exists an integrable function $I(v)$ such that $\frac{\partial}{\partial v}F(v;\tau) \leq I(v)$ over $(c,d) \times (-\rho',\rho')^m$, where ρ' is a strictly positive number not larger than ρ.

(iii) For all $1 \leq k \leq m$, $\frac{\partial}{\partial \tau_k}F(v;\tau)$ is bounded over $(c,d)\times(-\rho'',\rho'')^m$, where ρ'' is a strictly positive number not larger than ρ.

[7] In Appendix 3, we refer to another uniqueness result from Lebrun (2006). Other uniqueness results can be found in Corollary 4 in Lebrun (1999), Theorem 1 in Lebrun (2006), and Appendix 6 in Lebrun (2004).

[8] The derivative at $v = d$ is a lefthand derivative.

When the partial derivatives $\frac{\partial}{\partial v}F(v;\tau) = f(v;\tau)$ and $\frac{\partial}{\partial \tau_l}F(v;\tau)$, $1 \leq l \leq m$, are continuous over $(c,d] \times (-\rho,\rho)^m$, $f(v;\tau)$ is strictly positive over the same set, and $f(d;\tau)$ is continuously differentiable with respect to τ in $(-\rho,\rho)^m$, extending F according to the equality $F(v;\tau) = 1 + f(d;\tau)(v-d)$, for all v in $(d, d+\zeta)$, satisfies D (i).

We now find when the examples from the previous section satisfy the assumptions above.

Example 1: Example 1 (Section 2) satisfies Assumption E if F and $H_1, ..., H_m$ are differentiable over $(c,d]$ with respective derivatives $f, h_1, .., h_n$ such that $f - \rho \sum_{k=1}^{m} |h_k|$ is locally bounded away from zero. From our remark after the statement of Assumption D, it satisfies D(i) under the same conditions.

Decreasing ρ if necessary, it satisfies Assumption U if, for example, over an interval $[c, c+\varepsilon]$, where $\varepsilon > 0$, $\frac{d}{dv}f, \frac{d}{dv}h_1, ..., \frac{d}{dv}h_m$ exist and are continuous and $F(v)\frac{d}{dv}f(v) - f(v)^2$ has a strictly negative maximum.

D (iii) follows immediately from the continuity of $H_1, ..., H_m$. For all $\boldsymbol{\tau}$ in $(-\rho/2, \rho/2)^{nm}$, $|f + \sum_{k=1}^{m} \tau_k h_k|$ is not larger than $f + \frac{\rho}{2}\sum_{k=1}^{m}|h_k|$, which is integrable since f and h_k, $1 \leq k \leq n$, are, and D (ii) is satisfied.

Example 2: Example 2 (Section 2) satisfies Assumptions E and D (i) if F and $H_1, ..., H_m$ are differentiable over $(c,d]$ with derivatives $f, h_1, ..., h_m$ such that $f/F - \rho\sum_{k=1}^{m}|h_k|/H_k$ is locally bounded away from zero. If $H_1, ..., H_m$, and $F\left(\prod_{k=1}^{m} H_k\right)^{-\rho}$ are strictly log-concave in an interval $(c, c+\varepsilon)$, where $\varepsilon > 0$, it satisfies Assumption U.

We now show that D (ii) and D (iii) are satisfied if $H_1, ..., H_m$ are bounded. The derivative with respect to v of $F(.;\tau_1, ..., \tau_m)$ is equal to $F(v)\prod_{k=1}^{m} H_k(v)^{\tau_k}\left\{\frac{d}{dv}\ln F + \sum_{k=1}^{m}\tau_k \frac{d}{dv}\ln H_k\right\}$. Since the expression between braces is not smaller than $\frac{d}{dv}\ln F+\rho\sum_{k=1}^{m}\left|\frac{d}{dv}\ln H_k\right|$ and thus[9] than $2\frac{d}{dv}\ln F$, the derivative is not smaller than $2f(v)\prod_{k=1}^{m}H_k(v)^{\tau_k}$. Consequently, D (ii) is satisfied with $\rho'=\rho/2$ and $I(v)=2Kf(v)$, where K is a upper bound of $\prod_{k=1}^{m}H_k(v)^{\tau_k}$, for v in $[c,d]$ and $(\tau_1,...,\tau_m)$ in $[-\rho/2,\rho/2]^m$.

For all $1\leq l\leq m$ and all $(v;\tau_1,...,\tau_m)$ in $(c,d]\times(-\rho/2,\rho/2)^m$, the derivative with respect to τ_l of $F(.;\tau_1,...,\tau_m)$ is equal to $F(v)(\ln H_l(v))\prod_{k=1}^{m}H_k(v)^{\tau_k}$ or, equivalently:

$$\left(H_l(v)^{\rho/2}\ln H_l(v)\right)\left(F(v)H_l(v)^{-\rho/2}\prod_{\substack{k=1\\k\neq l}}^{m}H_k(v)^{\tau_k}\right).$$

From the properties of the logarithm, the first factor above is bounded. Since the second factor is a cumulative distribution function, it is also bounded. Example 2 thus satisfies D (iii) with $\rho''=\rho/2$.

Theorem 1 below is the main result of this section.

Theorem 1: *Let Assumptions E, U, and D be satisfied. Then:*

(i) For all $\boldsymbol{\tau}=(\tau^1,...,\tau^n)$ *in* $(-\rho,\rho)^{nm}$*, there exists one and only one equilibrium of the first-price auction with n-tuple of value distributions* $(F(.;\tau^1),...,F(.;\tau^n))$.

(ii) For all v *in* $(c,d]$ *and* $1\leq i\leq n$*, bidder i's equilibrium bid* $\beta_i(v;\boldsymbol{\tau})$*, interim expected payoff* $P_i(v;\boldsymbol{\tau})$*, and ex-ante expected payoff* $P_i(\boldsymbol{\tau})$ *are differentiable with respect to* $\boldsymbol{\tau}$ *at* $\boldsymbol{\tau}=\mathbf{0}$ *and the values of the partial derivatives with respect to the parameters are as in Appendix 2.*

(iii) For all $1\leq i\leq n$*, the auctioneer's revenues* $R^F(\boldsymbol{\tau})$ *and* $R^S(\boldsymbol{\tau})$ *are differentiable and equal to the first-order at* $\boldsymbol{\tau}=\mathbf{0}$.

[9] Because $\frac{d}{dv}\ln F-\rho\sum_{i=1}^{m}\left|\frac{d}{dv}\ln H_i\right|\geq 0$.

Proof: See Appendix 3.

We briefly outline the proof of Theorem 1. Consider the three-bidder case. For all vector of parameters $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$, the bid functions $\beta_1(.;\boldsymbol{\tau}), \beta_2(.;\boldsymbol{\tau}), \beta_3(.;\boldsymbol{\tau})$ form the unique equilibrium if and only if there exists $c < \eta < d$ such that their inverses $\alpha_1(.;\boldsymbol{\tau}), \alpha_2(.;\boldsymbol{\tau}), \alpha_3(.;\boldsymbol{\tau})$ satisfy the following system of differential equations with boundary conditions:

$$\frac{d\ln F(\alpha_1(b;\boldsymbol{\tau});\tau^1)}{dv} = \frac{1}{2}\left\{\frac{-1}{\alpha_1(b;\boldsymbol{\tau})-b}+\frac{1}{\alpha_2(b;\boldsymbol{\tau})-b}+\frac{1}{\alpha_3(b;\boldsymbol{\tau})-b}\right\}, \quad (5)$$

$$\frac{d\ln F(\alpha_2(b;\boldsymbol{\tau});\tau^2)}{dv} = \frac{1}{2}\left\{\frac{1}{\alpha_1(b;\boldsymbol{\tau})-b}-\frac{1}{\alpha_2(b;\boldsymbol{\tau})-b}+\frac{1}{\alpha_3(b;\boldsymbol{\tau})-b}\right\}, \quad (6)$$

$$\frac{d\ln F(\alpha_3(b;\boldsymbol{\tau});\tau^3)}{dv} = \frac{1}{2}\left\{\frac{1}{\alpha_1(b;\boldsymbol{\tau})-b}+\frac{1}{\alpha_2(b;\boldsymbol{\tau})-b}-\frac{1}{\alpha_3(b;\boldsymbol{\tau})-b}\right\}; \quad (7)$$

$$\alpha_1(c;\boldsymbol{\tau}) = \alpha_2(c;\boldsymbol{\tau}) = \alpha_3(c;\boldsymbol{\tau}) = c; \quad (8)$$

$$\alpha_1(\eta;\boldsymbol{\tau}) = \alpha_2(\eta;\boldsymbol{\tau}) = \alpha_3(\eta;\boldsymbol{\tau}) = d. \quad (9)$$

η in (9) is the common maximum of the equilibrium bid functions.

Once the differentiability with respect to the bid is established (see Lebrun 1999), the equations (5-7) follow immediately from the first-order conditions. For example, the first-order condition of bidder 1's maximization problem $\max_b (v_1 - b) F(\alpha_2(b;\boldsymbol{\tau});\tau^2) F(\alpha_3(b;\boldsymbol{\tau});\tau^3)$ when $v_1 = \alpha_1(b)$ gives (10) below. (11) and (12) are bidders 2 and 3's first-order conditions.

$$0 + \frac{d\ln F(\alpha_2(b;\boldsymbol{\tau});\tau^2)}{dv} + \frac{d\ln F(\alpha_3(b;\boldsymbol{\tau});\tau^3)}{dv} = \frac{1}{\alpha_1(b)-b}, \quad (10)$$

$$\frac{d\ln F(\alpha_1(b;\boldsymbol{\tau});\tau^1)}{dv} + 0 + \frac{d\ln F(\alpha_3(b;\boldsymbol{\tau});\tau^3)}{dv} = \frac{1}{\alpha_2(b)-b}, \quad (11)$$

$$\frac{d\ln F(\alpha_1(b;\boldsymbol{\tau});\tau^1)}{dv} + \frac{d\ln F(\alpha_2(b;\boldsymbol{\tau});\tau^2)}{dv} + 0 = \frac{1}{\alpha_3(b)-b}. \quad (12)$$

Adding up (11) and (12) and subtracting (10), we find (5). The other equations are obtained similarly.

We cannot apply the standard theorems regarding the differentiability with respect to parameters of the solution of a differential system with initial condition to (5-7) and (8) because the Lipschitz condition does not hold true at (8), where the denominators in (5-7) vanish. However, it does at (9). If we knew the exact value of η as a function of the parameters, we could infer its differentiability and then apply the standard theorems. Unfortunately, all we know is that it is uniquely determined. We now show how to circumvent this difficulty.

From Lemma A2-2 in Lebrun (1997) or Lemma A1-1 in Lebrun (2006), the derivatives in (5-7) are strictly positive. By multiplying the system (5-7) by the inverse of (5), we find the equivalent system:

$$\frac{d}{dq}\gamma_1(q;\boldsymbol{\tau}) = \frac{1}{q}\frac{2}{\frac{-1}{F_1^{-1}(q;\tau^1)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_2^{-1}(\lambda_{21}(q;\boldsymbol{\tau});\tau^2)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_3^{-1}(\lambda_{31}(q;\boldsymbol{\tau});\tau^3)-\gamma_1(q;\boldsymbol{\tau})}}, \quad (13)$$

$$\frac{d}{dq}\lambda_{21}(q;\boldsymbol{\tau}) = \frac{\lambda_{21}(q;\boldsymbol{\tau})}{q}\frac{\frac{1}{F_1^{-1}(q;\tau^1)-\gamma_1(q;\boldsymbol{\tau})}-\frac{1}{F_2^{-1}(\lambda_{21}(q;\boldsymbol{\tau});\tau^2)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_3^{-1}(\lambda_{31}(q;\boldsymbol{\tau});\tau^3)-\gamma_1(q;\boldsymbol{\tau})}}{\frac{-1}{F_1^{-1}(q;\tau^1)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_2^{-1}(\lambda_{21}(q;\boldsymbol{\tau});\tau^2)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_3^{-1}(\lambda_{31}(q;\boldsymbol{\tau});\tau^3)-\gamma_1(q;\boldsymbol{\tau})}}, \quad (14)$$

$$\frac{d}{dq}\lambda_{31}(q;\boldsymbol{\tau}) = \frac{\lambda_{31}(q;\boldsymbol{\tau})}{q}\frac{\frac{1}{F_1^{-1}(q;\tau^1)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_2^{-1}(\lambda_{21}(q;\boldsymbol{\tau});\tau^2)-\gamma_1(q;\boldsymbol{\tau})}-\frac{1}{F_3^{-1}(\lambda_{31}(q;\boldsymbol{\tau});\tau^3)-\gamma_1(q;\boldsymbol{\tau})}}{\frac{-1}{F_1^{-1}(q;\tau^1)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_2^{-1}(\lambda_{21}(q;\boldsymbol{\tau});\tau^2)-\gamma_1(q;\boldsymbol{\tau})}+\frac{1}{F_3^{-1}(\lambda_{31}(q;\boldsymbol{\tau});\tau^3)-\gamma_1(q;\boldsymbol{\tau})}}, \quad (15)$$

where $\gamma_1(.;\boldsymbol{\tau})$ is bidder 1's bid as a function of his value quantile[10], that is, $\gamma_1(q;\boldsymbol{\tau}) = \beta_1\left(F_1^{-1}(q;\tau^1);\boldsymbol{\tau}\right)$ and $\lambda_{21}, \lambda_{31}$ are the functions that link bidder 1's value quantile with bidders 2 and 3's value quantiles at which they submit the same bid, that is, for example, $\lambda_{21}(q;\boldsymbol{\tau})$ is equal to $F_2\left(\alpha_2\left(\beta_1\left(F_1^{-1}(.;\tau^1);\boldsymbol{\tau}\right);\boldsymbol{\tau}\right);\tau^2\right)$. The initial condition (9) is then equivalent to the set of equalities (16) and (17) below:

$$\gamma_1(1;\boldsymbol{\tau}) = \eta, \quad (16)$$

$$\lambda_{21}(1;\boldsymbol{\tau}) = \lambda_{31}(1;\boldsymbol{\tau}) = 1. \quad (17)$$

[10] Working with quantiles as the variables avoids imposing Lipschitz conditions on the density functions.

The part (17) of the initial conditions regarding the functions λ_{21} and λ_{31} is now independent of the parameters.

Consider, for example, the function λ_{21}. Since the initial condition (17) does not depend on $\boldsymbol{\tau}$, its derivative with respect to any combination of the parameters vanishes. Moreover, if we differentiate the equation (14) for $\frac{d}{dq}\lambda_{21}(q;\boldsymbol{\tau})$ with respect to the parameters and use the differentiability with respect to (q,τ^i), which Assumption D (i) implies, of $F^{-1}(q,\tau^i)$, the functions $\gamma_1(q;\boldsymbol{\tau})$ and $\lambda_{31}(q;\boldsymbol{\tau})$ disappear. In fact, these functions enter the same way the numerator and denominator of the R.H.S.'s ratio, which equals one for any symmetric case and, in particular, for $\boldsymbol{\tau}=\mathbf{0}$.

Although no general explicit expression exists for the maximum bid η, explicit bounds (see Lemma A3-2, Appendix 3) can be obtained from Lebrun (1997, 1999). These bounds and Assumption D (iii) guarantee that the difference ratio in the computation of the derivative of η in the initial condition (16) stays bounded, thereby entitling us to differentiate (14) with respect to the parameters. We thus obtain and uniquely solve a completely determined differential equation with initial condition where the derivative at $\boldsymbol{\tau}=\mathbf{0}$ of $\lambda_{21}(q;\boldsymbol{\tau})$ with respect to the parameters is the only unknown function.

In Appendix 3, we actually prove in general the joint differentiability of the functions such as $\lambda_{21},\lambda_{31}$ with respect to the vector $\boldsymbol{\tau}$ of parameters and the bid b. This joint differentiability and the definition of these functions imply the differentiability with respect to the parameters of bidder 1's interim probability of winning–the product of $F_2(\alpha_2(\beta_1(v_1;\boldsymbol{\tau});\boldsymbol{\tau});\tau^2)$ with $F_3(\alpha_3(\beta_1(v_1;\boldsymbol{\tau});\boldsymbol{\tau});\tau^3)$. The differentiability of bidder 1's bid $\beta_1(v_1;\boldsymbol{\tau})$ then comes from the envelope theorem, which links it to bidder 1's probability of winning. Thanks to Assumption D (ii), taking the expectation of the winner's bid gives expected revenues that are also differentiable with respect to the parameters.

4. Further Applications

In Section 2, we showed how our differentiability results imply properties of auction revenues. Here, we show two examples of applications that pertain to stochastic shifts of the value distributions in the first-price auction. The function $F(v;\tau)$ depends on a one-dimensional parameter τ in $(-\rho,\rho)$ such that, for all $\tau' > \tau$, $F(v;\tau')$ strictly dominates $F(v;\tau)$ in the sense of the conditional stochastic dominance, that is, the reverse hazard rate of $F(v;\tau')$ is larger than $F(v;\tau)$'s or, equivalently, $\frac{d}{dv}\ln F(v;\tau') > \frac{d}{dv}\ln F(v;\tau)$. It implies that the ratio $\frac{F(v;\tau')}{F(v;\tau)}$ is strictly increasing. Bidder i becomes "stronger" in this sense by, in the first application below, engaging in value enhancing investments and, in the second application, allowing more members in his cartel.

4.1 Application 1

In the models we consider in this application and the next, the cross second-order derivatives $\frac{\partial^2}{\partial v \partial \tau}\ln F(v;0)$ and $\frac{\partial^2}{\partial \tau \partial v}\ln F(v;0)$ exist and are equal. Since, from our assumption of stochastic dominance, $\frac{\partial^2}{\partial \tau \partial v}\ln F(v;0)$ is nonnegative, all the formulas we obtain from the previous section for the first-order effects (see Appendix 2) are consistent (as they should!) with the existing literature. For example, we have $\frac{\partial}{\partial \tau^j}\beta_i(v;\mathbf{0}) \geq 0$, $\frac{\partial}{\partial \tau^j}F(\alpha_j(b;\mathbf{0});0) \leq 0$, $\frac{\partial}{\partial \tau^j}P_i(v;\mathbf{0}) \leq \frac{\partial}{\partial \tau^j}P_j(v;\mathbf{0}) \leq 0$, which are consistent with the increase of bidder i's bid function, the stochastic increase of bidder j's bid distribution, and the decrease of the bidders' interim expected payoffs Lebrun (1998)[11] describes as consequences of a stochastic increase of bidder j's value distribution[12]. Since bidder j's value is more likely to be high, his exante expected

[11] Lebrun (1998) proves these results under the assumption that the bidders can be divided into two groups, such that the value distributions of the bidders within a group are identical. By constructing a counterexample, Lebrun (2002, Proposition 1) shows that these results cannot be extended to asymmetric settings with more than two groups of bidders.

[12] These inequalities are also consistent with some properties in Corollary 3 of Lebrun

payoff may improve, despite the other bidders' more aggressive bidding. Nevertheless, it may not. Indeed, Thomas (1997) provides a discrete analytical example where bidder j's exante expected payoff actually decreases after a stochastic improvement of his value distribution. Through numerical computations, Arozamena and Cantillon (2004) provide an example with interval supports where, after becoming stronger, bidder j also sees his expected payoff go down. Here, thanks to our explicit formulas for the first-order effects, it is simple to analytically generate examples where $\frac{\partial}{\partial \tau^j} P_j(\mathbf{0}) < 0$. Moreover, as we state in Corollary 1 below, such examples may be constructed as stochastic changes of any differentiable absolutely continuous distribution whose density does not go to infinity too fast at c.

Corollary 1[13]:

(i) Let $F(v;\tau)$ in (1) satisfy Assumptions E,U, and D with $m = 1$. Let also $F(v;\tau)$ be strictly increasing in τ for the conditional stochastic dominance and let $\frac{\partial^2}{\partial v \partial \tau} \ln F(v;0)$ and $\frac{\partial^2}{\partial \tau \partial v} \ln F(v;0)$ exist and be equal, for all v in $(c,d]$. If the inequality (18) below holds true:

$$\int_c^d F(v;0)^{n-1} K(v) \frac{\partial}{\partial \tau} \ln F(v;0)\, dv < 0, \quad (18)$$

where $K(v) = 1 - \frac{2n-1}{n-1} F(v;0) - \frac{\int_c^d F(w;0)^{n-1} dw}{F(v;0)^{n-1}} f(v)$, for all v in $(c,d]$, then we have $\frac{\partial}{\partial \tau^i} P_i(\mathbf{0}) < 0$, for all $1 \leq i \leq n$.

(ii) Let F be an absolutely continuous cumulative distribution function that is differentiable with a derivative f locally bounded away from zero over

(1999) according to which, within any equilibrium: a bidder bids higher than another bidder whose value distribution his distribution conditionally dominates (for the two-bidder case, see also Proposition 3.3 in Maskin and Riley, 2000); and the same first-order relation of stochastic dominance holds true between bidders' bid distributions as between their value distributions. See Application 2 below for another example of a link between first-order comparative statics and previously known properties of strategies within the same equilibrium.

[13] We prove in Appendix 4 the more general Corollary 1', which allows c to be a mass point.

$(c,d]$ and such that F is strictly log-concave over an interval $(c,c+\varepsilon)$, with $\varepsilon>0$, and $(v-c)\,f(v)$ tends towards zero as v tends towards c.

Then, there exists $F(v;\tau)$ that satisfies Assumptions E,U,D, is increasing in τ for the conditional stochastic dominance, and is such that $F(v;0)=F(v)$ and $\frac{\partial}{\partial\tau^i}P_i(\mathbf{0})<0$, for all $1\leq i\leq n$.

Proof: See Appendix 4.

For example, from Corollary 1, there exist stochastic improvements of a bidder's uniform value distribution $F(v)=v$ over $[0,1]$ that make him worse off in the first-price auction. From the construction in the proof of Corollary 1, the function $F(v;\tau)=v\exp\left\{-\tau(1-v)^{\theta}\right\}$, where $\theta>4$, describes such improvements. Starting from the uniform symmetric model, because of the indirect strategic effect through the change of the other bidders' equilibrium strategies, bidder i would not improve his value distribution to $v\exp\left\{-\tau(1-v)^{\theta}\right\}$, for a small positive τ, even if he could do so at no direct cost.

4.2 Application 2

In this application, $F(v;\tau)$ is the value distribution of a cartel of $1+\tau$ bidders. When bidders do not exert any positive or negative effect on their fellow cartel members, $F(v;\tau)$ is simply the distribution of the maximum of the members' values, that is, when the values are independently and identically distributed according to $F(v;0)$: $F(v;\tau)=F(v;0)^{1+\tau}$, $\frac{\partial\ln}{\partial v}F(v;\tau)=(1+\tau)\frac{\partial\ln}{\partial v}F(v;0)$, and $\frac{(\partial\ln)^2}{\partial\tau\partial v}F(v;\tau)=1$. For all $k>0$, $F(v;0)^{1+\tau}$ can as well be interpreted as the value distribution of a cartel that counts $k/\tau+k$ members, each with the value distribution $F(v;0)^{\tau/k}$. Waehrer (1997) proves that, in any equilibrium of the first-price auction among such cartels, a cartel has a smaller per-member average payoff than any smaller cartel.

Since Example 2 in Sections 2 and 3 encompasses this model, the equilibrium is differentiable around the symmetric case–of equal-size cartels–when Assumptions E,U, and D are satisfied. Assume this is the case. Then, the derivative $\frac{\partial}{\partial \tau^i} AP_i(\mathbf{0})$ of cartel i's average payoff with respect to τ^i, or with respect to its size $1+\tau^i$, must not exceed its derivative $\frac{\partial}{\partial \tau^j} AP_i(\mathbf{0})$ with respect to τ^j, for all $j \neq i$, that is, $\frac{\partial}{\partial \tau^i} AP_i(\mathbf{0}) \leq \frac{\partial}{\partial \tau^j} AP_i(\mathbf{0})$. Otherwise, after a transfer of $d\tau$ bidders from cartel j to cartel i, cartel i's average payoff would be larger than cartel j's by an amount equal to $\left(\frac{\partial}{\partial \tau^i} AP_i(\mathbf{0}) - \frac{\partial}{\partial \tau^j} AP_i(\mathbf{0})\right) d\tau > 0$, which would contradict Waehrer (1999)'s result. Here, from our explicit formulas for the partial derivatives, we prove Corollary 2 below, according to which the strict inequality $\frac{\partial}{\partial \tau^i} AP_i(\mathbf{0}) < \frac{\partial}{\partial \tau^j} AP_i(\mathbf{0})$ actually holds.

Transfers of bidders into a cartel have one obvious detrimental direct effect on the cartel's average payoff–the increase of the number of members–and two beneficial direct effects–the stochastic increase of its value distribution and the stochastic decrease of its competitors' value distributions. An additional detrimental strategic or indirect effect is also present: the other cartels bid more aggressively. From Corollary 2 below, the detrimental effects outweigh the beneficial ones and cause the expanding cartel's average payoff to decrease.

When, because of diseconomies of scale in the cartel size, $F(v;0)^{1+\tau}$ dominates $F(v;\tau)$ for $\tau > 0$, we have $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0) \leq 1$. This new adverse direct effect contribute, by slowing the increase of the expanding cartel's value distribution and the decrease of the competing shrinking cartels' value distributions, to make accepting transfers unattractive[14]. We have Corollary 2 below: this intuition is correct.

Corollary 2[15]: *Let $F(v;\tau)$ in (1) satisfy Assumptions E,U, and D with*

[14] It is straightforward to extend Waehrer (1999)'s result, which compares average payoffs within the same equilibrium, from the assumption $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;\tau) = 1$ to the assumption $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;\tau) \leq 1$, for all $(v;\tau)$.

[15] In Appendix 4, we prove Corollary 2', which, since it allows c to be a mass point, is more general than Corollary 2.

$m = 1$ and $\rho < 1$. Let $F(v;\tau)$ be strictly increasing in τ for the conditional stochastic dominance and let $\frac{\partial^2}{\partial v \partial \tau} \ln F(v;0)$ and $\frac{\partial^2}{\partial \tau \partial v} \ln F(v;0)$ exist and be equal, for all v in $(c,d]$. Let $AP_i(\boldsymbol{\tau})$ be defined as follows:

$$AP_i(\boldsymbol{\tau}) = \frac{P_i(\boldsymbol{\tau})}{1+\tau^i},$$

for all $1 \leq i \leq n$ and $\boldsymbol{\tau}$ in $(-\rho,\rho)^n$. Then, if $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0) \leq 1$, for all v in $(c,d]$, we have:

$$\frac{\partial}{\partial \tau^i} AP_i(\mathbf{0}) \leq \frac{\partial}{\partial \tau^j} AP_i(\mathbf{0}), \quad (19)$$

for all $i \neq j$. Moreover, if there exists $\varepsilon > 0$ such that $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0) > 0$, for all v in $(d-\varepsilon, d)$, the inequality in (19) is strict.

Proof: See Appendix 4.

Thus, in the first-price auction, admitting transfers into a cartel can only make sense when the additional bidders are numerous or strong enough, that is, $d\tau$ is large enough, or when there are strong enough synergies, that is, $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0)$ is large enough. To illustrate this point, assume that, in the case $\frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;\tau) = 1$ without synergies, two cartels of five bidders each form out of 10 bidders whose values are identically distributed over $[0,1]$ according to $F(v) = v^{1/2}$. From the standard formulas for the symmetric case, one can easily compute that the per-member average payoff of each cartel is 0.0238. From the numerical estimations[16] in Marshall, Meurer, Richard, and Stromquist (1994), if only one bidder is transferred from one cartel to the other, the average payoff of the cartel with the new bidder drops to 0.0233 (while the average payoff of the other cartel jumps to 0.0261). However, if two more bidders are transferred, the average payoff of the larger cartel, now

[16] Marshall *et al* (1994) actually consider a total of five bidders with uniformly distributed values. The figures for our 6-4 and 8-2 splits come from their figures for the 3-2 and 4-1 splits.

counting eight members, becomes 0.0283.

5. Conclusion

We showed that, under joint differentiability of the equilibrium with respect to the parameters, the equality in the symmetric case between the revenues of two anonymous auction mechanisms is to the first-order in the asymmetry parameters. We proved this differentiability for the second-price and first-price auctions. For the first-price auction, we obtained general formulas for the first-order effects of a change of parameters on the equilibrium bids and payoffs. As examples of other applications, we showed how to analytically generate continuous cases where shifting a bidder's distribution towards higher values lowers his payoff, and we proved that, without strong synergies, it is unprofitable for a cartel to become slightly larger than its competitors through transfers of members.

Appendix 1

Proof of Proposition 1 (Section 2): Let D be the m-dimensional linear space spanned by the vectors $(\tau^1, ..., \tau^n)$ such that $\tau^i = \tau^j$, for all $1 \leq i, j \leq n$. Let $D_\perp$ be its orthogonal complement. By definition, the dimension of $D_\perp$ is $nm - m = (n-1)\,m$ and we have $R^{nm} = D \oplus D_\perp$. The space $D_\perp$ is the set of vectors $(\tau^1, ..., \tau^n)$ such that $\sum_{i=1}^n \tau^i = 0$. It is equal to the direct sum $\oplus_{l=2}^m V_l$, where V_l is the m-dimensional space spanned by the vectors $(\tau^1, ..., \tau^n)$ such that $\tau^k = 0$, for all $k \neq 1, l$, and $\tau^l = -\tau^1$.

Let N be equal to M or S. Let $dR^N(\mathbf{0})$ be the differential of R^N at $\mathbf{0}$, considered as a linear function from R^{nm} to R. Let l between 2 and n and let $\boldsymbol{\tau}$ be an element of V_l. If the derivative of R^N at $\mathbf{0}$ in the direction of $\boldsymbol{\tau}$ was different from zero, the values $R^N(\lambda\boldsymbol{\tau})$ and $R^N(-\lambda\boldsymbol{\tau})$ would be different, for all number λ close enough to zero. However, this is impossible, since $\lambda\boldsymbol{\tau}$ and $-\lambda\boldsymbol{\tau}$ are equal up to a permutation of their n components and R^N is symmetric. Consequently, $dR^N(\mathbf{0})$ vanishes over $D_\perp$.

The derivative of R^M at $\mathbf{0}$ in any direction $\left(\delta^1, ..., \delta^n\right)$ is thus equal to its derivative in the direction of the orthogonal projection $proj_D\left(\delta^1, ..., \delta^n\right)$ of $\left(\delta^1, ..., \delta^n\right)$ over D. Since, by assumption, R^M and R^S coincide over D, it is also equal to the derivative of R^S in the same direction. Proposition 1 (ii) then follows from the equality $proj_D\left(\delta^1, ..., \delta^n\right) = \left(\sum_{i=1}^n \delta^i/n, ..., \sum_{i=1}^n \delta^i/n\right)$,, which can be proved by checking that the difference between $\left(\delta^1, ..., \delta^n\right)$ and $\left(\sum_{i=1}^n \delta^i/n, ..., \sum_{i=1}^n \delta^i/n\right)$ is orthogonal to any vector $(\tau, ..., \tau)$ in D.

Let $\Delta(\boldsymbol{\tau})$ be the difference $R^M(\boldsymbol{\tau}) - R^S(\boldsymbol{\tau})$ and let $d\Delta(\mathbf{0})$ be its differential at $\mathbf{0}$. Since, $d\Delta(\mathbf{0})$ is equal to $dR^M(\mathbf{0}) - dR^S(\mathbf{0})$, it also vanishes over $D_\perp$. By assumption, Δ vanishes over D and hence so does its differential $d\Delta(\mathbf{0})$. We have proved that the differential $d\Delta(\mathbf{0})$ vanishes everywhere over R^{nm}. Proposition 1 (i) follows. ||

Appendix 2

The formulas in 2., 3., 4. below hold true if $\frac{\partial}{\partial \tau_l} F(v; 0)$ is differentiable with respect to v in $(c, d]$, for all $1 \leq l \leq m$, and in 5., 6. if $\frac{\partial^2}{\partial v \partial \tau_l} \ln F(v; 0)$ and $\frac{\partial^2}{\partial \tau_l \partial v} \ln F(v; 0)$ exist and are equal. Without these additional assumptions, the formulas are less compact.

1. $\lambda_{ji}(.; \boldsymbol{\tau}) = F(.; \tau^j) \circ \alpha_j(.; \boldsymbol{\tau}) \circ \beta_i(.; \boldsymbol{\tau}) \circ F(.; \tau^i)^{-1}$:

$$\begin{aligned}
\frac{\partial}{\partial \tau_l^j} \lambda_{ji}(q; \mathbf{0}) &= -\frac{\partial}{\partial \tau_l^i} \lambda_{ji}(q; \mathbf{0}) \\
&= (n-1)\, q \left(\int_c^{F^{-1}(q;0)} F(w; 0)^{n-1} dw \right)^{n-1} . \\
&\quad \int_q^1 \frac{p^{n-2} \frac{\partial}{\partial \tau_l} F\left(F^{-1}(p; 0); 0\right)}{f\left(F^{-1}(p; 0); 0\right) \left(\int_c^{p^{-1}(q;0)} F(w; 0)^{n-1} dw \right)^n} dp; \quad \text{(A2.1)} \\
\frac{\partial}{\partial \tau_l^h} \lambda_{ji}(q; \mathbf{0}) &= 0, \text{ for all } h \neq i, j.
\end{aligned}$$

2. $\varphi_{ki}(.;\boldsymbol{\tau}) = \alpha_k(.;\boldsymbol{\tau}) \circ \beta_i(.;\boldsymbol{\tau})$:

$$\begin{aligned}
\frac{\partial}{\partial \tau_l^j}\varphi_{ji}(v;\mathbf{0}) &= -\frac{\partial}{\partial \tau_l^i}\varphi_{ji}(v;\mathbf{0}) \\
&= \frac{\int_v^d \frac{d\frac{\partial}{\partial \tau_l}\ln F(w;0)}{\left(\int_c^w F(x;0)^{n-1}dx\right)^{n-1}}}{\frac{f(v;0)}{F(v;0)\left(\int_c^v F(x;0)^{n-1}dx\right)^{n-1}}}; \\
\frac{\partial}{\partial \tau_l^h}\varphi_{ji}(v;\mathbf{0}) &= 0, \text{ for all } h \neq i, j.
\end{aligned}$$

3. Equilibrium bid functions[17]:

$$\begin{aligned}
&\frac{\partial}{\partial \tau_l^j}\beta_i(v;\mathbf{0}) \\
&= \frac{n-1}{n}\left\{\begin{array}{c} \frac{\int_c^v\left(\int_c^w F(z;0)^{n-1}dz\right)d\frac{\partial}{\partial \tau_l}\ln F(w;0)}{F(v;0)^{n-1}} \\ +\frac{\left(\int_c^v F(w;0)^{n-1}dw\right)^n}{F(v;0)^{n-1}}\int_v^d \frac{d\frac{\partial}{\partial \tau_l}\ln F(y;0)}{\left(\int_c^y F(w;0)^{n-1}dw\right)^{n-1}} \end{array}\right\};
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial \tau_l^i}\beta_i(v;\mathbf{0}) \\
&= \frac{n-1}{n}\left\{\begin{array}{c} \frac{\int_c^v\left(\int_c^w F(z;0)^{n-1}dz\right)d\frac{\partial}{\partial \tau_l}\ln F(w;0)}{F(v;0)^{n-1}} \\ -(n-1)\frac{\left(\int_c^v F(w;0)^{n-1}dw\right)^n}{F(v;0)^{n-1}}\int_v^d \frac{d\frac{\partial}{\partial \tau_l}\ln F(y;0)}{\left(\int_c^y F(w;0)^{n-1}dw\right)^{n-1}} \end{array}\right\}.
\end{aligned}$$

[17] The derivative of $\beta_i(v;\boldsymbol{\tau})$ at $\boldsymbol{\tau} = \mathbf{0}$ in the direction $\left(\delta^1, ..., \delta^n\right)$, where $\delta_k^j = 0$ if $k \neq j$ and $\delta_k^k = 1$, is equal to $\sum_{j=1}^n \frac{\partial}{\partial \tau_j^j}\beta_i(v;\mathbf{0})$. Substituting in this expression the formulas below, applied to Example 1 (Section 2), easily gives (after some rearranging and one integration by parts) the first-order effect that Fibich and Gavious (2003) find the perturbations (3) have on the bid functions.

4. Interim expected payoffs:

$$\begin{aligned}
&\frac{\partial}{\partial \tau_l^i} P_i(v;\mathbf{0}) \\
= \quad & -\frac{n-1}{n}\left\{\begin{array}{c} \int_c^v \left(\int_c^w F(z;0)^{n-1}\,dz\right) d\frac{\partial}{\partial \tau_l}\ln F(w;0) \\ +\left(\int_c^v F(w;0)^{n-1}\,dw\right)^n \int_v^d \frac{d\frac{\partial}{\partial \tau_l}\ln F(y;0)}{\left(\int_c^y F(w;0)^{n-1}dw\right)^{n-1}} \end{array}\right\};
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial \tau_l^j} P_i(v;\mathbf{0}) \\
= \quad & \frac{\partial}{\partial \tau_l^i} P_i(v;\mathbf{0}) \\
& +(n-1)\left(\int_c^v F(w;0)^{n-1}\,dw\right)^n \int_v^d \frac{F(y;0)^{n-1}\, d\frac{\partial}{\partial \tau_l}\ln F(y;0)}{\left(\int_c^y F(w;0)^{n-1}\,dw\right)^n}.
\end{aligned}$$

5. Ex-ante expected payoffs:

$$\begin{aligned}
&\frac{\partial}{\partial \tau_l^j} P_i(\mathbf{0}) \\
= \quad & \frac{n-1}{n}\int_c^d \int_c^v F(z;0)^{n-1}\frac{\partial}{\partial \tau_l}\ln F(z;0)\,dz dF(v;0) \\
& +\frac{1}{n}\int_c^d \frac{\partial}{\partial \tau_l}\ln F(v;0)\int_c^v F(z;0)^{n-1}\,dz dF(v;0) \\
& +\frac{1}{n}\int_c^d \left(\int_v^d \frac{d\frac{\partial \ln}{\partial \tau_l} F(y;0)}{\left(\int_c^y F(z;0)^{n-1}\,dz\right)^{n-1}}\right)\left(\int_c^v F(z;0)^{n-1}\,dz\right)^n dF(v;0);
\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial}{\partial \tau_l^i} P_i(\mathbf{0}) \\
= \; & \frac{\partial}{\partial \tau_l^j} P_i(\mathbf{0}) \\
& + \int_c^d \left(\frac{(\partial \ln)^2}{\partial \tau_l \partial v} F(v;0) - 1 \right) \left(\int_c^v F(w;0)^{n-1} dw \right) dF(v;0) \\
& - \int_c^d \left(\int_v^d \frac{d \frac{\partial \ln}{\partial \tau_l} F(y;0)}{\left(\int_c^y F(z;0)^{n-1} dz \right)^{n-1}} \right) \left(\int_c^v F(z;0)^{n-1} dz \right)^n dF(v;0) \\
& + \int_c^d \int_c^v F(w;0)^{n-1} dw dF(v;0).
\end{aligned}
$$

6. Cartels' per-bidder average ex-ante expected payoffs:

$$
\begin{aligned}
\frac{\partial}{\partial \tau_l^j} AP_i(\mathbf{0}) &= \frac{\partial}{\partial \tau_l^j} P_i(\mathbf{0}); \\
\frac{\partial}{\partial \tau_l^i} AP_i(\mathbf{0}) &= \frac{\partial}{\partial \tau_l^i} P_i(\mathbf{0}) - \int_c^d \int_c^v F(w;0)^{n-1} dw dF(v;0).
\end{aligned}
$$

Appendix 3

Throughout Appendices 3 and 4, we maintain the assumption from the main text that $F(.;\tau)$ ((1), Section 2) be absolutely continuous over $[c,d]$.

Assumption EU' below pertains to the case of a mass point at c. It is satisfied by Example 1 (Section 2) if $F(c) - \rho \sum_{k=1}^m |H_k(c)| > 0$ and by Example 2 (Section 2) if $F(c), H_1(c), ..., H_m(c) > 0$.

Assumption EU': $F(.;\tau)$ has a mass point at c and is differentiable–with respect to v–over $[c,d]$ with a derivative[18]–the density function $f(.;\tau)$–that is bounded away from zero over this interval, for all τ in $(-\rho,\rho)^m$.

[18] The derivative at c is a righthand derivative.

In this appendix, we prove Theorem 1' below, which extends Theorem 1 (Section 3)[19].

Theorem 1':

(i) Let either Assumptions E and U or Assumption EU' be satisfied. Then, for all $\boldsymbol{\tau} = (\tau^1, ..., \tau^n)$ *in* $(-\rho, \rho)^{nm}$*, there exists one and only one equilibrium of the first-price auction with n-tuple of value distributions* $(F(.;\tau^1), ..., F(.;\tau^n))$.

(ii) Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied. Then, for all v *in* $(c, d]$ *and* $1 \leq i \leq n$*, bidder i's equilibrium bid* $\beta_i(v; \boldsymbol{\tau})$*, interim expected payoff* $P_i(v; \boldsymbol{\tau})$*, and exante expected payoff* $P_i(\boldsymbol{\tau})$ *are differentiable with respect to* $\boldsymbol{\tau}$ at $\boldsymbol{\tau} = \mathbf{0}$ *and the values of the partial derivatives with respect to the parameters are as in Appendix 2.*

(iii) Let Assumptions E, U, and D be satisfied. Then, for all $1 \leq i \leq n$*, the auctioneer's revenues* $R^F(\boldsymbol{\tau})$ *and* $R^S(\boldsymbol{\tau})$ *are differentiable and equal to the first-order at* $\boldsymbol{\tau} = \mathbf{0}$.

We divide the proof of Theorem 1' into several lemmas. First, we have

[19] Further generalizations are possible. For example, Theorem 1 (i) and (ii) can be proved under Assumption D (i), either Assumptions E, U or Assumption EU', and the following assumption:

There exist $0 < \sigma < \rho$, functions l and u defined over $(c, d] \times (0, \sigma)$, and an integrable function k defined over $(c, d]$ such that:

1. For all (v, σ') in $(c, d] \times (0, \sigma)$ and τ^i, τ^j in $(-\sigma', \sigma')^m$,

$$\begin{aligned} & l(v; \sigma') \\ \leq \; & F\left(v; \tau^i\right) \min_{w \in [v,d]} \frac{F\left(w; \tau^j\right)}{F\left(w; \tau^i\right)} \\ \leq \; & F\left(v; \tau^i\right) \max_{w \in [v,d]} \frac{F\left(w; \tau^j\right)}{F\left(w; \tau^i\right)} \\ \leq \; & u(v; \sigma'); \end{aligned}$$

2. $\frac{\left|l\left(v;\sigma'\right) - F(v)\right|}{\sigma'}$ and $\frac{\left|u\left(v;\sigma'\right) - F(v)\right|}{\sigma'}$ are not larger than $k(v)$ over $(c, d] \times (0, \sigma)$.

Lemma A3-1 below.[20]

Lemma A3-1: *Let either Assumptions E and U or Assumption EU' be satisfied. Then, for all* $\boldsymbol{\tau} = (\tau^1, ..., \tau^n)$ *in* $(-\rho, \rho)^{nm}$*:*

(i) There exists an "essentially" unique Bayesian equilibrium $(\beta_1(.;\tau), ..., \beta_n(.;\tau))$ *of the first-price auction with value distributions* $F_1 = F(.;\tau^1), ..., F_n = F(.;\tau^n)$*. This equilibrium is pure and there exists* $c < \eta < d$ *such that the inverse bid functions* $\alpha_1 = \beta_1^{-1}, ..., \alpha_n = \beta_n^{-1}$ *exist, are strictly increasing, and form a solution over* $(c, \eta]$ *of the system of differential equations (A3.1) below–considered in the domain* $D = \{(b, \alpha_1, .., \alpha_n) \in R^{n+1} | c, b < \alpha_i \leq d, \text{ for all } 1 \leq i \leq n\}$*–and satisfy the boundary conditions (A3.2-3.3):*

$$\frac{d \ln F\left(\alpha_k(b;\boldsymbol{\tau});\tau^k\right)}{db} = \frac{1}{n-1}\left\{\frac{-(n-2)}{\alpha_k(b;\boldsymbol{\tau})-b} + \sum_{\substack{l=1\\ l\neq k}}^{n} \frac{1}{\alpha_l(b;\boldsymbol{\tau})-b}\right\}, \text{ for all } 1 \leq k \leq n; \text{ (A3.1)}$$

$$\begin{aligned}\alpha_k(\eta) &= d, \text{ for all } 1 \leq k \leq n; \text{ (A3.2)}\\ \alpha_i(c) &= c, \text{ for all but at most one } i \text{ between } 1 \text{ and } n; \text{ (A3.3)}\end{aligned}$$

and $\beta_k(v;\boldsymbol{\tau}) = OUT$*, for all* v *in* $[c, \alpha_k(c;\boldsymbol{\tau})]$ *and all* $1 \leq k \leq n$*. Moreover,* $\frac{d}{db}\alpha_k(b;\boldsymbol{\tau}) > 0$*, for all* $1 \leq k \leq n$ *and all* b *in* $(c, \eta]$*.*

(ii) For all $1 \leq i \leq n$*, the functions* $\lambda_{ji}(.;\boldsymbol{\tau}) = F(.;\tau^j) \circ \alpha_j(.;\boldsymbol{\tau}) \circ \beta_i(.;\boldsymbol{\tau}) \circ F(.;\tau^i)^{-1}$*, with* $1 \leq j \leq n$ *and* $j \neq i$*, and* $\gamma_i(.;\boldsymbol{\tau}) = \beta_i(.;\boldsymbol{\tau}) \circ F(.;\tau^i)^{-1}$ *are differentiable (with respect to the first argument) over* $(F(\alpha_i(c;\boldsymbol{\tau});\tau^i), 1]$ *and form a solution of the system (A3.4-3.5)–considered in the domain* D_i*–*

[20] In Lemma A3-1 (i), bidding OUT means remaining out of the auction. "Essential" uniqueness refers to uniqueness for values in $(c, d]$. The only possible indeterminacy for an essentially unique equilibrium is at the lowest value c, where some bidders may choose OUT, c, or randomize between the two. Here, we assume that every bidder k submits OUT over $[c, \alpha_k(c;\boldsymbol{\tau})]$. All equilibria can be characterized as in Lebrun (1997) by replacing this assumption by the following condition:

If there exists j such that $\alpha_j(c;\boldsymbol{\tau}) > c$, then $\beta_i(c;\boldsymbol{\tau}) = OUT$, for all $i \neq j$, and $\beta_j(v_j;\boldsymbol{\tau}) = c$, for all v_j in $(c, \alpha_j(c;\boldsymbol{\tau})]$.

with initial condition (A3.6):

$$D_i = \left\{ \begin{array}{c} \left(q, (\lambda_{ji})_{j\neq i}, \gamma_i\right) | F(c;\tau^i) < q \leq 1,\ F(c;\tau^j) < \lambda_{ji} \leq 1, \\ \text{and } \gamma_i < F^{-1}(q;\tau^i), F^{-1}(\lambda_{ji};\tau^j), \text{ for all } j \neq i \end{array} \right\}$$

$$\begin{aligned} & \frac{d}{dq}\lambda_{ji}(q;\boldsymbol{\tau}) \\ = \; & \frac{\lambda_{ji}(q;\boldsymbol{\tau})}{q} \frac{\frac{-(n-2)}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1 \\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}}{\frac{-(n-2)}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1 \\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}} \text{(A3.4)} \end{aligned}$$

$$\frac{d}{dq}\gamma_i(q;\boldsymbol{\tau}) = \frac{1}{q} \frac{n-1}{\frac{-(n-2)}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1 \\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}};\text{(A3.5)}$$

$$\lambda_{ji}(1;\boldsymbol{\tau}) = 1, \gamma_i(1;\boldsymbol{\tau}) = \eta.\text{(A3.6)}$$

(iii) For all $1 \leq i \leq n$ and v in $(c,d]$, (A3.7) below holds true:

$$\beta_i(v;\boldsymbol{\tau}) = v - \frac{\int_c^v \prod_{\substack{k=1 \\ k\neq i}}^n F\left(\varphi_{ki}(w;\boldsymbol{\tau});\tau^k\right) dw}{\prod_{\substack{k=1 \\ k\neq i}}^n F\left(\varphi_{ki}(v;\boldsymbol{\tau});\tau^k\right)}, \text{ (A3.7)}$$

where $\varphi_{ki}(.;\boldsymbol{\tau}) = \alpha_k(.;\boldsymbol{\tau}) \circ \beta_i(.;\boldsymbol{\tau})$.

(iv) If $\tau^1 = ... = \tau^n$, then:

$$\beta_i(v;\boldsymbol{\tau}) = v - \frac{\int_c^v F(w;\tau^i)^{n-1} dw}{F(v;\tau^i)^{n-1}},$$

for all $1 \leq i \leq n$ and v in $(c,d]$.

Proof: The existence of an equilibrium in (i) follows from Theorem 2 in Lebrun (1999) and its characterization from Theorem 1 in Lebrun (1999).

The uniqueness in (i) under E and U follows from Corollary 1 in Lebrun (1999). The uniqueness under EU' follows from Theorem 1 in Lebrun (2006).

(ii) follows from Lemma A2-5 in Lebrun (1997) or from Lemma A1-1 in Lebrun (2006). An application, standard in auction theory, of the envelope theorem gives (iii) ((iii) also follows from Lemma A2-6 in Lebrun 1997). (iv) follows from Corollary 3 (v) in Lebrun (1999). ||

Although not explicitly proved in Lebrun (1997), Lemma A3-2 below can easily be derived from the proof of its Lemma A2-3. We provide the proof for the sake of completeness.

Lemma A3-2: *Let either Assumptions E and U or Assumption EU' be satisfied. Then, for all τ in $(-\rho,\rho)^{nm}$, all v in $(c,d]$, and all $1 \leq i,j \leq n$, we have:*

$$F\left(v;\tau^i\right)\min_{w\in[v,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)} \leq F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right) \leq F\left(v;\tau^i\right)\max_{w\in[v,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)},$$

where $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$ is equal to $\alpha_j\left(\beta_i\left(v;\boldsymbol{\tau}\right);\boldsymbol{\tau}\right)$.

Proof: Subtracting the equation in (A3.1) for $\frac{d\ln F\left(\alpha_i(b;\boldsymbol{\tau});\tau^i\right)}{db}$ from the equation for $\frac{d\ln F\left(\alpha_j(b;\boldsymbol{\tau});\tau^j\right)}{db}$, we find:

$$\frac{d\ln F\left(\alpha_j\left(b;\boldsymbol{\tau}\right);\tau^j\right)}{db}-\frac{d\ln F\left(\alpha_i\left(b;\boldsymbol{\tau}\right);\tau^i\right)}{db}=\frac{1}{\alpha_i\left(b;\boldsymbol{\tau}\right)-b}-\frac{1}{\alpha_j\left(b;\boldsymbol{\tau}\right)-b}. \quad \text{(A3.8)}$$

Let u be in $(c,d]$ and let $z>0$ such that $z<\min_{w\in[u,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)}$.

Define y in $[u,d]$ as follows: $y=\inf\left\{w \text{ in } [u,d]\,|zF\left(w;\tau^i\right)\geq F\left(c;\tau^j\right)\right\}$, with the convention $d=\inf\varnothing$. Since $\varphi_{ji}\left(w;\tau\right)\geq c$, for all w, we have $zF\left(w;\tau^i\right)\leq F\left(\varphi_{ji}\left(w;\boldsymbol{\tau}\right);\tau^j\right)$, for all w in (u,y). Suppose v in $(y,d]$ is such that $zF\left(v;\tau^i\right)=F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right)$. Then, $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)>c$. From (A3.8), we have:

$$\frac{d\ln F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^{j}\right)}{dv} = \frac{d\ln F\left(v;\tau^{i}\right)}{dv} + \frac{d}{dv}\beta_{i}\left(v;\boldsymbol{\tau}\right)\left\{\frac{1}{v-\beta_{i}\left(v;\boldsymbol{\tau}\right)} - \frac{1}{\varphi_{ji}\left(v;\boldsymbol{\tau}\right)-\beta_{i}\left(v;\boldsymbol{\tau}\right)}\right\}. \text{(A3.9)}$$

By definition of z, we have $z < \frac{F\left(v;\tau^{j}\right)}{F\left(v;\tau^{i}\right)}$ and thus $zF\left(v;\tau^{i}\right) < F\left(v;\tau^{j}\right)$. Consequently, $\varphi_{ji}\left(v;\boldsymbol{\tau}\right) < v$. Since $\frac{d}{dv}\ln zF\left(v;\tau^{i}\right) = \frac{d}{dv}\ln F\left(v;\tau^{i}\right)$, (A3.9) then implies $\frac{d\ln F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau^{j}\right)}{dv} < \frac{d}{dv}\ln zF\left(v;\tau^{i}\right)$. Moreover, from the definition of z, $zF\left(d;\tau^{i}\right) = z < 1 = F\left(d;\tau^{j}\right) = F\left(\varphi_{ji}\left(d;\boldsymbol{\tau}\right);\tau^{j}\right)$. From a variant of Lemma 2 in Milgrom and Weber (1982), we obtain $zF\left(w;\tau^{i}\right) \leq F\left(\varphi_{ji}\left(w;\boldsymbol{\tau}\right);\tau^{j}\right)$, for all w in $[y,d]$, hence in $[u,d]$, and, in particular, $zF\left(u;\tau^{i}\right) \leq F\left(\varphi_{ji}\left(u;\boldsymbol{\tau}\right);\tau^{j}\right)$. Finally, making z tend towards $\min_{w\in[u,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)}$, we find $F\left(u;\tau^{i}\right)\min_{w\in[u,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)} \leq F\left(\varphi_{ji}\left(u;\boldsymbol{\tau}\right);\tau^{j}\right)$. The other inequality can be proved similarly. ||

Lemma A3-3: *Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied. Let* $\eta\left(\boldsymbol{\tau}\right)$ *be the common maximum of the equilibrium bid functions* $\beta_{1}\left(.;\boldsymbol{\tau}\right),...,\beta_{n}\left(.;\boldsymbol{\tau}\right)$, *for all* $\boldsymbol{\tau}$ *in* $(-\rho,\rho)^{nm}$. *Then, there exists* K *and* $0<\rho'<\rho$ *such that* $\frac{|\eta(0)-\eta(\boldsymbol{\tau})|}{|\boldsymbol{\tau}|} \leq K$, *for all* $\boldsymbol{\tau}$ *in* $(-\rho',\rho')^{nm}$.

Proof: From Lemma A3-2, we have:

$$F\left(v;\tau^{i}\right)\min_{w\in[v,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)} \leq F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^{j}\right) \leq F\left(v;\tau^{i}\right)\max_{w\in[v,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)},$$

where $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$ is equal to $\alpha_{j}\left(\beta_{i}\left(v;\boldsymbol{\tau}\right);\boldsymbol{\tau}\right)$, for all $1\leq i,j\leq n$, $\boldsymbol{\tau}$ in

$(-\rho,\rho)^{nm}$, and v in $(c,d]$. From Assumption D (iii), we then have:

$$\begin{aligned} F(v;\tau_i)\left(1-\frac{M|\tau_j-\tau_i|}{F(v;\tau_i)}\right) &= F(v;\tau_i)\min_{w\in[v,d]}\left(1-\frac{M|\tau_j-\tau_i|}{F(w;\tau_i)}\right) \\ &\leq F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau_j\right)\leq \\ F(v;\tau_i)\max_{w\in[v,d]}\left(1+\frac{M|\tau_j-\tau_i|}{F(w;\tau_i)}\right) &= F(v;\tau_i)\left(1+\frac{M|\tau_j-\tau_i|}{F(v;\tau_i)}\right), \quad \text{(A3.10)} \end{aligned}$$

and thus:

$$F(v;0)-M(|\tau_j|+2|\tau_i|)\leq F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau_j\right)\leq F(v;0)+M(|\tau_j|+2|\tau_i|),$$

where M is an upper bound of $\frac{\partial}{\partial\tau_1}F(v;\tau),...,\frac{\partial}{\partial\tau_m}F(v;\tau)$ over $(c,d]\times(-\rho'',\rho'')^m$, for all $1\leq i,j\leq n$, $\boldsymbol{\tau}$ in $(-\rho'',\rho'')^{nm}$, and v in $(c,d]$.

From Lebrun (1999) (or the envelope theorem), we have, for $1\leq i\leq n$, $\eta(\boldsymbol{\tau})=d-\int_c^d\prod_{\substack{j=1\\j\neq i}}^{n}F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau_j\right)dv$ and thus, from (A3.10):

$$\begin{aligned} &\int_c^d\frac{F(v;0)^{n-1}-\left(F(v;0)+M(|\tau_j|+2|\tau_i|)\right)^{n-1}}{|\boldsymbol{\tau}|}dv \\ \leq\ &\frac{\eta(0)-\eta(\boldsymbol{\tau})}{|\boldsymbol{\tau}|} \\ \leq\ &\int_c^d\frac{F(v;0)^{n-1}-\max\left(0,F(v;0)-M(|\tau_j|+2|\tau_i|)\right)^{n-1}}{|\boldsymbol{\tau}|}dv, \quad \text{(A3.11)} \end{aligned}$$

for all $(v;\boldsymbol{\tau})$ in $(c,d]\times(-\rho'',\rho'')^{nm}$. From the mean value theorem, for all v in $(c,d]$, there exists x between $F(v;0)$ and $F(v;0)+M(|\tau_j|+2|\tau_i|)$, such that $\frac{F(v;0)^{n-1}-(F(v;0)+M(|\tau_j|+2|\tau_i|))^{n-1}}{|\boldsymbol{\tau}|}$ is equal to $-(n-1)x^{n-2}\frac{M(|\tau_j|+2|\tau_i|)}{|\boldsymbol{\tau}|}$. Since $0\leq x\leq 1+3\rho''M$ and $0\leq\frac{|\tau_j|+2|\tau_i|}{|\boldsymbol{\tau}|}\leq 3$, there exists a finite K' such that the L.H.S. of the first inequality in (A3.11) is not smaller than K'. Similarly, there exists a finite K'' such that the R.H.S of the second inequality is not larger than K'', for all $\boldsymbol{\tau}$ in $(-\rho'',\rho'')^{nm}$. The lemma follows. ||

Lemma A3-4: *Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied. Then,* $\alpha_i(c;\boldsymbol{\tau})$ *is continuous with respect to* $\boldsymbol{\tau}$ *at* $\boldsymbol{\tau}=\mathbf{0}$.

Proof: From Lemma A3-3, $\eta(\boldsymbol{\tau})$ is a continuous function of $\boldsymbol{\tau}$ at $\boldsymbol{\tau}=\mathbf{0}$. From Lemma A3-1 (i) and from the continuity, under our assumptions, of the solution of a differential system with respect to the parameters and to the value of the solution at the initial condition, we know that for all b in the interior $(c,\eta(\mathbf{0}))$ of the definition domain of $\alpha_1(.,\mathbf{0})=...=\alpha_n(.,\mathbf{0})$ and for all $\varepsilon>0$, there exists $\delta>0$ such that, for all $1\leq i\leq n$, $\alpha_i(.,\boldsymbol{\tau})$ is defined at b and $|\alpha_i(b,\boldsymbol{\tau})-\alpha_i(b,\mathbf{0})|\leq\varepsilon$ if $|\boldsymbol{\tau}|<\delta$. Consequently, for all b in $(c,\eta(\mathbf{0}))$, $\limsup_{\boldsymbol{\tau}\to\mathbf{0}}\alpha_i(c,\boldsymbol{\tau})\leq\alpha_i(b,\mathbf{0})\leq b$. By making b tend towards c, we find $\limsup_{\boldsymbol{\tau}\to\mathbf{0}}\alpha_i(c,\boldsymbol{\tau})\leq c$. Since $\alpha_i(c,\boldsymbol{\tau})$ is never smaller than c, we have $\lim_{\boldsymbol{\tau}\to\mathbf{0}}\alpha_i(c,\boldsymbol{\tau})=c$ and Lemma A3-4 is proved. ||

Lemma A3-5: *Let Assumption D (i) be satisfied and let F be extended over* $(c,d+\zeta)\times(-\rho,\rho)^m$, *with* $\zeta>0$, *as in D (i). Then, there exists* $\zeta'>0$ *such that* $F^{-1}(q;\tau)$ *exists and is (jointly) continuously differentiable with respect to* $(q;\tau)$ *over* $\{(q,\tau)\,|\tau\in(-\rho,\rho)^m, q\in(F(c;\tau),1+\zeta')\}$, *and, for all* (q,τ) *in this set and all* $1\leq l\leq m$, *we have:*

$$
\begin{aligned}
\frac{\partial}{\partial q}F^{-1}(q;\tau) &= \frac{1}{f(F^{-1}(q;\tau);\tau)},\\
\frac{\partial}{\partial \tau_l}F^{-1}(q;\tau) &= \frac{-\frac{\partial}{\partial\tau_l}F(F^{-1}(q;\tau);\tau)}{f(F^{-1}(q;\tau);\tau)}.
\end{aligned}
$$

Proof: It suffices to apply the inverse function theorem to the function $\mathcal{F}$ such that $\mathcal{F}(v,\tau)=(F(v;\tau),\tau)$, for all $(v;\tau)$ in $(c,d+\zeta)\times(-\rho,\rho)^m$. ||

Lemma A3-6: *Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied and let F be extended over* $(c,d+\zeta)\times(-\rho,\rho)^m$, *with* $\zeta>0$, *as in D (i). Let* $\boldsymbol{\tau}(\pi)$

be a continuously differentiable function from $(-1,1)$ to $(-\rho,\rho)^{nm}$ such that $\boldsymbol{\tau}(0)=\mathbf{0}$. Then, for all sequence $(\Delta\pi_k)_{k\geq 1}$ of strictly positive numbers converging towards 0, there exists a subsequence $(\Delta\pi_{k_t})_{t\geq 1}$ such that, for all q in an interval $(F(c;0),1+\zeta')$ with $\zeta'>0$ and all $1\leq i\neq j\leq n$, $\lim_{t\to+\infty}\frac{\lambda_{ji}(q;\mathbf{0})-\lambda_{ji}\left(q;\boldsymbol{\tau}\left(\Delta\pi_{k_t}\right)\right)}{\Delta\pi_{k_t}}$ exists, is finite, and is equal to $\overline{\lambda}_{ji}(q)$ below:

$$\begin{aligned}\overline{\lambda}_{ji}(q) &= q\left(\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}\,dw\right)^{n-1}. \\ &\left\{\begin{array}{c}\sum_{l=1}^{m}(n-1)\int_q^1\frac{p^{n-2}\frac{\partial}{\partial\tau_l}F\left(F^{-1}(p;0);0\right)}{f(F^{-1}(p;0);0)\left(\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw\right)^n}dp \\ \left(\frac{d}{d\pi}\tau_l^j(0)-\frac{d}{d\pi}\tau_l^i(0)\right).\end{array}\right\}.\text{(A3.12)}\end{aligned}$$

Proof: For all $\boldsymbol{\tau}$ in $(-\rho,\rho)^{nm}$, let $\eta(\boldsymbol{\tau})$ be the common maximum of the equilibrium bid functions. From Lemma A3-3, there exists a subsequence $(\Delta\pi_{k_t})_{t\geq 1}$ such that $\lim_{t\to+\infty}\frac{\eta(\mathbf{0})-\eta\left(\boldsymbol{\tau}\left(\Delta\pi_{k_t}\right)\right)}{\Delta\pi_{k_t}}$ exists and is finite. Let χ be this limit. For all $1\leq i\neq j\leq n$ and $q>F(\alpha_i(c;\mathbf{0});0)$, we may assume, from Lemma A3-1 (ii) and Lemma A5-2 in Appendix 5, that $\overline{\lambda}_{ji}(q)=\lim_{t\to+\infty}\frac{\lambda_{ji}(q;\mathbf{0})-\lambda_{ji}\left(q;\boldsymbol{\tau}\left(\Delta\pi_{k_t}\right)\right)}{\Delta\pi_{k_t}}$ and $\overline{\gamma}_i(q)=\lim_{t\to+\infty}\frac{\gamma_i(q;\mathbf{0})-\gamma_i\left(q;\boldsymbol{\tau}\left(\Delta\pi_{k_t}\right)\right)}{\Delta\pi_{k_t}}$ exist and form a solution of the linear differential system obtained from (A3.4-3.5) by differentiating it around its solution $\lambda_{ji}(.;\mathbf{0}),\gamma_i(.;\mathbf{0})$, and of the initial condition below:

$$\begin{aligned}\overline{\lambda}_{ji}(1) &= 0,\ j\neq i,\text{(A3.13)} \\ \overline{\gamma}_i(1) &= \chi.\end{aligned}$$

Differentiating (A3.4) with respect to π, setting $\pi=0$, using the equalities $\boldsymbol{\tau}(0)=\mathbf{0}$ and $\lambda_{ji}(q;\mathbf{0})=q$, for all q in the interval $(F(c;0),1+\zeta')$, where ζ' is from Lemma A3-5, and rearranging, we find that the coefficients of $\overline{\lambda}_{hi}$,

$h \neq j, i$, and $\overline{\gamma}_i$ cancel out in $\overline{\lambda}_{ji}$'s equation, and we have:

$$\begin{aligned}\frac{d}{dq}\overline{\lambda}_{ji}(q) &= \frac{n-1}{F^{-1}(q;0)-\gamma_i(q;0)}\sum_{l=1}^{m}\frac{\partial}{\partial \tau_l}F^{-1}(q;0)\left(\frac{d}{d\pi}\tau_l^j(0)-\frac{d}{d\pi}\tau_l^i(0)\right)\\ &\quad+\left\{\frac{\partial}{\partial q}F^{-1}(q;0)\frac{n-1}{F^{-1}(q;0)-\gamma_i(q;0)}+\frac{1}{q}\right\}\overline{\lambda}_{ji}(q).\end{aligned}$$

From Lemma A3-1 (iv)[21] and Lemma A3-5, we obtain:

$$\begin{aligned}\frac{d}{dq}\overline{\lambda}_{ji}(q) &= -\frac{(n-1)\,q^{n-1}}{f\left(F^{-1}(q;0);0\right)\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw}.\\ &\quad\sum_{l=1}^{m}\frac{\partial}{\partial \tau_l}F\left(F^{-1}(q;0);0\right)\left(\frac{d}{d\pi}\tau_l^j(0)-\frac{d}{d\pi}\tau_l^i(0)\right)\\ &\quad+\left\{\frac{(n-1)\,q^{n-1}}{f\left(F^{-1}(q;0);0\right)\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw}+\frac{1}{q}\right\}\overline{\lambda}_{ji}(q).\qquad\text{(A3.14)}\end{aligned}$$

Using, for example, the method of "variation of constants," we easily find that the unique solution of (A3.13) and (A3.14) is (A3.12). ||

Lemma A3-7: *Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied and let F be extended over* $(c, d+\zeta)\times(-\rho,\rho)^m$*, with* $\zeta > 0$, as in D (i). *Then, for all* $1 \leq i \neq j \leq n$ *and q in an interval* $(F(c;0), 1+\zeta')$*, with* $\zeta' > 0$, $\lambda_{ji}(q;\boldsymbol{\tau})$ *is differentiable at* $(q;\mathbf{0})$ *and its partial derivatives are as in Appendix 2.*

Proof: Let $\boldsymbol{\tau}(\pi)$ be a continuously differentiable function from $(-1,1)$ to $(-\rho,\rho)^{nm}$ such that $\boldsymbol{\tau}(0)=\mathbf{0}$. From Lemma A3-6, $\lim_{\Delta\pi\to 0}\frac{\lambda_{ji}(q;\mathbf{0})-\lambda_{ji}(q;\boldsymbol{\tau}(\Delta\pi))}{\Delta\pi}$ exists and is equal to $\overline{\lambda}_{ji}(q)$ in (A3.12), for all q in an open interval $(F(c;0), 1+\zeta')$, where $\zeta' > 0$. In fact, otherwise there would exist a sequence $(\Delta\pi_k)_{k\geq 1}$ such that the difference ratio would be bounded away from $\overline{\lambda}_{ji}(q)$, which would

[21] $\lambda_{ji}(q;\mathbf{0}) = q$, with $j \neq i$, and $\gamma_i(q;\mathbf{0}) = F^{-1}(q;0) - \frac{\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw}{q^{n-1}}$, obtained from Lemma A3-1 (iv) for q in $(F(c;0), 1]$, also describe the solution to (A3-4-A3-6) past 1 in $[1, 1+\zeta')$.

contradict Lemma A3-5. Consequently, $\left(\frac{d}{d\pi}\lambda_{ji}\left(q;\boldsymbol{\tau}\left(\pi\right)\right)\right)_{\pi=0}$ exists and is equal to $\overline{\lambda}_{ji}\left(q\right)$ in (A3.12), which is linear in $\overline{\tau}_l^k$.

The differentiability with respect to $\boldsymbol{\tau}$ at $(q,\mathbf{0})$ then follows from Lemma A5-3[22] in Appendix 5. Finally, the joint differentiability with respect to $(q,\boldsymbol{\tau})$ follows from Lemma A5-4 in Appendix 5. The formulas in Appendix 2 come from (A3.12). ||

Lemma A3-8:

(i) Let either Assumptions E and U or Assumption EU' be satisfied. Let also Assumptions D (i) and (iii) be satisfied and let F be extended over $(c,d+\zeta)\times(-\rho,\rho)^m$, *with* $\zeta>0$, as in D (i). *Then, for all* $1\leq i\neq j\leq n$ *and* v *in* $(c,d+\zeta)$, *the function* $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$, *the probability of winning* $\prod_{l\neq i}F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right)$, *the interim payoff* $P_i\left(v;\boldsymbol{\tau}\right)=\int_c^v\prod_{l\neq i}F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right)dw$, *and the bid function* $\beta_i\left(v;\boldsymbol{\tau}\right)$ *are differentiable with respect to* $(v,\boldsymbol{\tau})$ *at* $(v,\mathbf{0})$.[23] *Also, the exante payoff* $P_i\left(\boldsymbol{\tau}\right)=\int_c^d P_i\left(v;\boldsymbol{\tau}\right)dF_i\left(v;\tau^i\right)$ *is differentiable at* $\boldsymbol{\tau}=\mathbf{0}$. *Moreover, the formulas in Appendix 2 for the partial derivatives apply.*

(ii) Let Assumptions E, U, and D be satisfied. Then, the expected total surplus $\sum_{i=1}^n\int_c^d v_i\left(\prod_{l\neq i}F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right)\right)dF\left(v;\tau^i\right)$ *and the auctioneer's revenues* $R^F\left(\boldsymbol{\tau}\right)$ *are differentiable at* $\boldsymbol{\tau}=\mathbf{0}$.

Proof: (i) We first prove the differentiability of these functions. From the definitions, we have $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)=F^{-1}\left(\lambda_{ji}\left(F\left(v;\tau^i\right);\boldsymbol{\tau}\right);\tau^j\right)$, for all $v>\alpha_i\left(c;\boldsymbol{\tau}\right)$, and $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)=c$, for $v\leq\alpha_i\left(c;\boldsymbol{\tau}\right)$. The differentiability of $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$ and $\prod_{l\neq i}F\left(\varphi_{li}\left(v;\boldsymbol{\tau}\right);\tau^l\right)$ then follows from Lemmas A3-7, A3-5, and A3-4.

[22] Since we obtain the first-order effects from differential equations at the symmetric case, we need a local condition, such as Lemma A5-3, that is sufficient for differentiability. We could not apply more familiar global conditions like, for example, the existence and continuity of the partial derivatives everywhere in a neighborhood of the symmetric case.

[23] Obviously, $P_i\left(v;\boldsymbol{\tau}\right)$ and $\beta_i\left(v;\boldsymbol{\tau}\right)$ are the interim payoff and bid function only for v in $(c,d]$.

For all continuously differentiable function $\boldsymbol{\tau}(\pi)$ from $(-1,1)$ to $(-\rho,\rho)^{nm}$ such that $\boldsymbol{\tau}(0)=\mathbf{0}$, the integral in the equality (A3.15) below is differentiable with respect to π at $\pi=0$ because (by applying Lebesgue theorem of dominated convergence, for example) the function inside the integral is differentiable with respect to $\boldsymbol{\tau}$ at $\boldsymbol{\tau}=\mathbf{0}$ and because, as in the proof of Lemma A3-3, Assumptions D (i, iii) imply that the ratio $\frac{\left|F(v;\mathbf{0})^{n-1}-\prod_{l\neq i}F\left(\varphi_{li}(v;\boldsymbol{\tau});\tau^{l}\right)\right|}{|\boldsymbol{\tau}|}$ is bounded:

$$P_i\left(v;\boldsymbol{\tau}(0)\right)=\int_l^v\prod_{l\neq i}F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}(0)\right);\tau^j(0)\right)dw. \quad \text{(A3.15)}$$

Moreover, differentiation may be taken under the integral sign, and, from the linearity of the integral, $\left(\frac{d}{d\pi}P_i\left(v;\boldsymbol{\tau}(\pi)\right)\right)_{\pi=0}$ is a linear function of $\frac{d}{d\pi}\tau_k^l(0)$, $1\leq l\leq n, 1\leq k\leq m$. The differentiability of $P_i(v;\boldsymbol{\tau})$ and $\beta_i(v;\boldsymbol{\tau})$ then follows from Lemma A5-3 in Appendix 5 and Lemma A3-1 (iii).

Integration by parts gives:

$$P_i(\boldsymbol{\tau})=\int_c^d\prod_{l\neq i}F\left(\varphi_{li}(v;\boldsymbol{\tau});\tau^l\right)\left(1-F\left(v;\tau^i\right)\right)dv. \quad \text{(A3.16)}$$

The ratio $\left\{\prod_{l\neq i}F\left(\varphi_{li}(v;\boldsymbol{\tau});\tau^l\right)\left(1-F\left(v;\tau^i\right)\right)-F(v;0)^{n-1}\left(1-F(v;0)\right)\right\}/|\boldsymbol{\tau}|$ can be broken down as the sum of $\left(1-F\left(v;\tau^i\right)\right)\left\{\prod_{l\neq i}F\left(\varphi_{li}(v;\boldsymbol{\tau});\tau^l\right)-F(v;0)^{n-1}\right\}/|\boldsymbol{\tau}|$ and $F(v;0)^{n-1}\left(F(v;0)-F\left(v;\tau^i\right)\right)/|\boldsymbol{\tau}|$. Assumption D (iii) immediately implies that the absolute value of the second term is bounded over $(c,d]\times(-\rho'',\rho'')^{nm}$. Proceeding as in the proof of Lemma A3-3, it also implies that the absolute value of the first term is bounded. As in the previous paragraph, the differentiability of $P_i(\boldsymbol{\tau})$ at $\boldsymbol{\tau}=\mathbf{0}$ then follows from the differentiability of $\prod_{l\neq i}F\left(\varphi_{li}(v;\boldsymbol{\tau});\tau^l\right)\left(1-F\left(v;\tau^i\right)\right)$.

We next prove the formulas in Appendix 2. From the definition of $\varphi_{ji}(v;\boldsymbol{\tau})$, Lemma A3-5, and $\frac{\partial}{\partial q}\lambda_{ji}(F(v;0);\mathbf{0})=1$, we have:

$$\left(\frac{d}{d\pi}\varphi_{ji}(v;\boldsymbol{\tau}(\pi))\right)_{\pi=0}=\frac{1}{f(v;0)}\left\{\begin{array}{c}\left(\frac{d}{d\pi}\lambda_{ji}\left(F\left(v;\tau^{i}(\pi)\right);\boldsymbol{\tau}(\pi)\right)\right)_{\pi=0}\\+\sum_{l=1}^{m}\frac{\partial}{\partial\tau_l}F^{-1}(q;\tau)\left(\frac{d}{d\pi}\tau_l^i(0)-\frac{d}{d\pi}\tau_l^j(0)\right)\end{array}\right\}.\text{(A3.17)}$$

Formulas for the partial derivatives of $\varphi_{ji}(v;\boldsymbol{\tau})$ can then be obtained by substituting in the equality above its value from Lemma A3-7 and (A2.1) (Appendix 2) to $\left(\frac{d}{d\pi}\lambda_{ji}\left(F\left(v;\tau^{i}(\pi)\right);\boldsymbol{\tau}(\pi)\right)\right)_{\pi=0}$.

As we now show, under the assumption that $\frac{\partial}{\partial\tau_l}F(v;0)$ is differentiable with respect to v, for all $1\leq l\leq m$, it is possible to simplify these formulas somewhat. Using the equality $\frac{\partial}{\partial\tau_l}F\left(F^{-1}(p;0);0\right)/f\left(F^{-1}(p;0);0\right)=\frac{\partial}{\partial\tau_l}\ln F\left(F^{-1}(p;0);0\right)$ and changing the variable to $w=F^{-1}(p;0)$, we see that (A2.1) in Appendix 2 is equal to the product of $q\left(\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw\right)^{n-1}$ and the integral below:

$$\int_{F^{-1}(q;0)}^{d}(-1)\frac{\partial}{\partial\tau_l}\ln F(w;0)\,d\frac{1}{\left(\int_c^w F(x;0)^{n-1}dx\right)^{n-1}}.$$

Integrating by parts and using $\frac{\partial}{\partial\tau_l}\ln F(d;0)=0$, we find the following equivalent expression for $\frac{\partial}{\partial\tau_l^j}\lambda_{ji}(q;\mathbf{0})$:

$$\begin{aligned}\frac{\partial}{\partial\tau_l^j}\lambda_{ji}(q;\mathbf{0}) &= \frac{\partial}{\partial\tau_l}F\left(F^{-1}(q;0);0\right)\\ &\quad +q\left(\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw\right)^{n-1}\int_{F^{-1}(q;0)}^{d}\frac{d\frac{\partial}{\partial\tau_l}\ln F(w;0)}{\left(\int_c^w F(x;0)^{n-1}dx\right)^{n-1}}.\end{aligned}$$

Substituting these new expressions in (A3.17), we find the formulas for the partial derivatives of $\varphi_{ji}(v;\boldsymbol{\tau})$ in Appendix 2.

Differentiating, which, as we proved above, we may do, under the integral signs in (A3.7), (A3.15), and (A3.16) and using the expressions for the partial derivatives of φ_{ji}, we find the expressions in Appendix 2 for the partial derivatives of the bid functions and the interim and exante payoffs.

(ii) Since $R^F(\boldsymbol{\tau})$ is the difference between the expected total surplus and the sum $\sum_{i=1}^{n} P_i(\boldsymbol{\tau})$ of the bidder's payoffs, which is differentiable from (i), we will have proved the differentiability of the total surplus and $R^F(\boldsymbol{\tau})$ once we prove the differentiability of $\int_c^d v \left(\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}); \tau^j\right) \right) \frac{\partial}{\partial v} F(v; \tau^i)\, dv$, for all $1 \leq i \leq n$. Let $\boldsymbol{\tau}$ be a continuously differentiable function from $(-1,1)$ to $(-\min(\rho', \rho''), \max(\rho', \rho''))^{nm}$, where ρ' and ρ'' are from Assumptions D (ii,iii), such that $\boldsymbol{\tau}(0) = \mathbf{0}$. Let N be an upper bound of the absolute values of its partial derivatives, so that $\frac{|\boldsymbol{\tau}(\pi)|}{|\pi|} \leq N$. The ratio $\left\{ \int_c^d v \left(\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}(\pi)); \tau^j(\pi)\right) \right) \frac{\partial}{\partial v} F(v; \tau^i(\pi))\, dv - \int_c^d vF(v;0)^{n-1}\, dF(v;0) \right\} / \boldsymbol{\pi}$ is equal to the sum of the two following integrals: $\int_c^d v \frac{\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}(\pi)); \tau^j(\pi)\right) - F(v;0)^{n-1}}{\pi} \frac{\partial}{\partial v} F(v; \tau^i(\pi))\, dv$, $\int_c^d vF(v;0)^{n-1} \frac{\frac{\partial}{\partial v} F\left(v; \tau^i(\pi)\right) - \frac{\partial}{\partial v} F(v;0)}{\pi} dv$. From Assumptions D (ii, iii), the absolute value of the function inside the first integral is not larger than the integrable function $vNMI(v)$, where M is an upper bound of $\frac{\left| \prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}); \tau^j\right) - F(v;0)^{n-1} \right|}{|\boldsymbol{\tau}|}$, for all $(v; \boldsymbol{\tau})$ in $(c,d] \times (-\min(\rho', \rho''), \max(\rho', \rho''))^{nm}$, and I is the integrable function from Assumption D (ii). The existence of the limit of $\int_c^d v \frac{\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}(\pi)); \tau^j(\pi)\right) - F(v;0)^{n-1}}{\pi} \frac{\partial}{\partial v} F(v; \tau^i(\pi))\, dv$ then follows from the differentiability of $\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}); \tau^j\right)$, the continuity of $\frac{\partial}{\partial v} F(v; \tau^i)$, and, for example, the Lebesgue Theorem of dominated convergence. Moreover, from the linearity of the integral, it is equal to a linear function of $\frac{d}{d\pi} \tau_k^l(0)$, $1 \leq l \leq n, 1 \leq k \leq m$.

The limit of the second integral exists because $\int_c^d vF(v;0)^{n-1} \frac{\partial}{\partial v} F(v; \tau^i(\pi))\, dv$ is differentiable at $\pi = 0$. In fact, integrating by parts, it is equal to $d - \int_c^d F(v; \tau^i(\pi))\, d\left(vF(v;0)^{n-1}\right)$. Assumptions D (ii, iii), as in the paragraph above, and the continuous differentiability of $\tau^i(\pi)$ imply that the absolute value $\frac{\left|F(v;0)-F\left(v;\tau^i\right)\right|}{|\boldsymbol{\tau}(\pi)|}\frac{|\boldsymbol{\tau}(\pi)|}{|\pi|}\frac{\partial}{\partial v}\left(vF(v;0)^{n-1}\right)$ of the ratio in the definition of the derivative is bounded by an integrable function of v only. Consequently, $\int_c^d vF(v;0)^{n-1}\frac{\partial}{\partial v}F\left(v;\tau^i(\pi)\right)dv$ is differentiable at $\pi=0$, its derivative is equal to $\int_c^d\left[\frac{d}{d\pi}F\left(v;\tau^i(\pi)\right)\right]_{\pi=0}d\left(vF(v;0)^{n-1}\right)$ and, from the linearity of the integral, is a linear function of $\frac{d}{d\pi}\tau_k^l(0)$, $1\leq l\leq n, 1\leq k\leq m$. The differentiability of $R^F(\boldsymbol{\tau})$ at $\boldsymbol{\tau}=\mathbf{0}$ then follows from Lemma A5-3. The differentiability of $R^S(\boldsymbol{\tau})$ follows from Proposition 2. The rest of (ii) then follows from Proposition 1 (Section 2). ||

Proof of Theorem 1': Theorem 1' (i) follows from Lemma A3-1 (i). Lemma A3-8 (ii) and (iii) imply Theorem 1' (ii) and (iii). ||

Appendix 4

Corollary 1':

(i) For $m=1$, let $F(v;\tau)$ satisfy Assumption D and either Assumptions E and U or Assumption EU'. Let $F(v;\tau)$ be strictly increasing in τ for the conditional stochastic dominance and let $\frac{\partial^2}{\partial v\partial\tau}\ln F(v;0)$ and $\frac{\partial^2}{\partial\tau\partial v}\ln F(v;0)$ exist and be equal, for all v in $(c,d]$. If the inequality (18) holds true, then we have $\frac{\partial}{\partial\tau^i}P_i(\mathbf{0})<0$, for all $1\leq i\leq n$.

(ii) Let F be a cumulative distribution function that is absolutely continuous over $[c,d]$ and is differentiable with a derivative f locally bounded away from zero over $(c,d]$ and such that (ii.1) or (ii.2) below holds true:

(ii.1) F has an atom at c such that $F(c)<\frac{n-1}{2n-1}$ and the continuous extension of f exists and is strictly positive at c;

(ii.2) F is atomless, strictly log-concave over an interval $(c,c+\varepsilon)$ with $\varepsilon>0$, and such that $(v-c)f(v)$ tends towards zero as v tends towards c.

Then, there exists $F(v;\tau)$ that satisfies Assumption D and either Assumptions E and U or Assumption EU', is increasing in τ for the conditional stochastic dominance, and is such that $F(v;0) = F(v)$ and $\frac{\partial}{\partial \tau^i} P_i(\mathbf{0}) < 0$, for all $1 \leq i \leq n$.

Proof: From the formulas in Appendix 2, $\frac{\partial}{\partial \tau^i} P_i(\mathbf{0})$ is equal to the sum of the following terms: $\frac{n-1}{n} \int_c^d \int_c^v F(z;0)^{n-1} \frac{\partial}{\partial \tau} \ln F(z;0)\, dz dF(v;0)$, $\frac{1}{n} \int_c^d \frac{\partial}{\partial \tau} \ln F(v;0) \int_c^v F(z;0)\, dz dF(v;0)$, $\int_c^d \frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0) \left(\int_c^v F(w;0)^{n-1} dw\right) dF(v;0)$, and $-\frac{n-1}{n} \int_c^d \left(\int_v^d \frac{d \frac{\partial}{\partial \tau} \ln F(y;0)}{\left(\int_c^y F(z;0)^{n-1} dz\right)^{n-1}} \right) \left(\int_c^v F(z;0)^{n-1} dz\right)^n dF(v;0)$. Since the last term is nonnegative, $\frac{\partial}{\partial \tau^i} P_i(\mathbf{0})$ is not larger than the sum of the first three terms. Through integration by parts, the first and third terms are equal to $\frac{n-1}{n} \int_c^d F(v;0)^{n-1} (1 - F(v;0)) \frac{\partial}{\partial \tau} \ln F(v;0)\, dv$ and $- \int_c^d F(v;0)^{n-1} \frac{\partial}{\partial \tau} \ln F(v;0)\, dv - \int_c^d \frac{\partial}{\partial \tau} \ln F(v;0) \left(\int_c^v F(w;0)^{n-1} dw\right) dF(v;0)$ (for the third term, we use the equality $\frac{F(v;0)}{f(v;0)} \frac{\partial}{\partial v} \frac{\partial}{\partial \tau} \ln F(v;0) = \frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0)$). Substituting their values to these terms gives the inequality $\frac{\partial}{\partial \tau^i} P_i(\mathbf{0}) \leq \frac{n-1}{n} \int_c^d F(v;0)^{n-1} K(v) \frac{\partial}{\partial \tau} \ln F(v;0)\, dv$, where K is as defined in (i). The first statement of Corollary 1' then follows.

Under either of the assumptions (ii.1) and (ii.2), $\lim_{v \to c} \left(\frac{\int_c^v F(w)^{n-1} dw}{F(v)^{n-1}} \right) f(v) = \lim_{v \to c} \left(\frac{\int_c^v F(w)^{n-1} dw}{(v-c) F(v)^{n-1}} \right) (v-c) f(v) = 0$. Consequently, $\lim_{v \to c} K(v) = 1 - \frac{2n-1}{n-1} F(c) < 0$. There thus exists a strictly log-concave continuously differentiable function $\zeta(v)$ (close to zero over $(c+\delta, d)$, where δ is small and strictly positive) such that $\zeta(d) = 0$, $\zeta(v) < 0$, for all v in $[c,d)$, the derivative $\frac{d}{dv} \zeta(w)$ is strictly positive and bounded over $[c,d]$, and $\int_c^d F(v;0)^{n-1} K(v) \zeta(v)\, dv < 0$. It suffices then to define $F(v;\tau)$ over $(c,d] \times (-\rho, \rho)$ by the equality $F(v;\tau) = F(v) \exp\left\{ \frac{J}{\rho L} \zeta(v) \tau \right\}$, where J is a strictly positive lower bound of $\frac{d}{dv} \ln F(w)$ (that such a bound exists follows from $f(c), F(c) > 0$ under (ii.1) and from the log-concavity of F over $(c, c+\varepsilon)$ under (ii.2)) and L an upper bound of $\frac{d}{dv} \zeta(w)$ over $[c,d]$. ||

Corollary 2': *Same statement as Corollary 2 (Section 4), except that $F(v;\tau)$ may satisfy Assumption EU' instead of Assumptions E and U.*

Proof: Corollary 2' is an immediate consequence of the following equality, obtained from the formulas in Appendix 2:

$$\begin{aligned}
&\frac{\partial}{\partial \tau^j} AP_i(\mathbf{0}) - \frac{\partial}{\partial \tau^i} AP_i(\mathbf{0}) \\
= &\int_c^d \left(1 - \frac{(\partial \ln)^2}{\partial \tau \partial v} F(v;0)\right) \left(\int_c^v F(w;0)^{n-1} dw\right) dF(v;0) \\
&+ \int_c^d \left(\int_v^d \frac{d\frac{\partial \ln}{\partial \tau} F(y;0)}{\left(\int_c^y F(z;0)^{n-1} dz\right)^{n-1}}\right) \left(\int_c^v F(z;0)^{n-1} dz\right)^n dF(v;0).
\end{aligned}$$

||

Appendix 5

Lemma A5-1: *Let $(\pi_k, \eta_k)_{k \geq 1}$ be a sequence in $R \times R^n$ converging towards $(\overline{\pi}, \overline{\eta})$ and such that $\pi_k \neq \overline{\pi}$, for all $k \geq 1$. If $\lim_{k \to +\infty} \frac{\eta_k - \overline{\eta}}{\pi_k - \overline{\pi}}$ exists and is finite, then there exists a subsequence $\left(\pi_{k_m}, \eta_{k_m}\right)_{m \geq 1}$ and a continuously differentiable function $\widetilde{\eta}$ from $(\overline{\pi} - 1, \overline{\pi} + 1)$ to R^n, such that $\widetilde{\eta}(\overline{\pi}) = \overline{\eta}$ and $\widetilde{\eta}(\pi_{k_m}) = \eta_{k_m}$, for all $m \geq 1$ such that $\pi_{k_m} \in (\overline{\pi} - 1, \overline{\pi} + 1)$.*

Proof: By considering a subsequence if necessary, we may assume that $(\pi_k)_{k \geq 1}$ is strictly monotonic. Assume, for example, that it is strictly decreasing (the proof is similar when it is strictly increasing). We first prove the lemma for $(\overline{\pi}, \overline{\eta}) = 0$. Let χ be equal to $\lim_{k \to +\infty} \frac{\eta_k}{\pi_k}$. Let k_1 be a value of the index such that $\pi_{k_1} \leq 1$ and $\left|\chi - \frac{\eta_{k_1}}{\pi_{k_1}}\right| \leq 1$. Assume k_m has been defined and $\left|\chi - \frac{\eta_{k_m}}{\pi_{k_m}}\right| \leq 1/m$. Then k_{m+1} is a value of the index such that $k_{m+1} > k_m$, $\left|\chi - \frac{\eta_{k_{m+1}}}{\pi_{k_{m+1}}}\right| \leq 1/(m+1)$, and $\left|\chi - \frac{\eta_{k_m} - \eta_{k_{m+1}}}{\pi_{k_m} - \pi_{k_{m+1}}}\right| \leq 2/m$. The last requirement can be satisfied because $(\pi_k, \eta_k)_{k \geq 1}$ tends towards 0 and $\left|\chi - \frac{\eta_{k_m}}{\pi_{k_m}}\right| \leq 1/m$.

By extracting a subsequence as in the previous paragraph if necessary, we may assume that $(\pi_k, \eta_k)_{k \geq 1}$ is such that $\left|\chi - \frac{\eta_k}{\pi_k}\right| \leq 1/k$ and $\left|\chi - \frac{\eta_k - \eta_{k+1}}{\pi_k - \pi_{k+1}}\right| \leq$

$2/k$, for all $k \geq 1$. Consider a step function σ from $(-1,1)$ to R^n such that $\sigma(\pi) = \frac{\eta_k - \eta_{k+1}}{\pi_k - \pi_{k+1}}$, for all π and k such that $\pi \in (\pi_{k+1}, \pi_k)$. Then approximate σ by a continuous function ζ from $(-1,1)$ to R^n such that $\int_{\pi_{k+1}}^{\pi_k} (\zeta(\pi) - \sigma(\pi))\, d\pi = 0$, for all $k \geq 1$. Such a function exists. In fact, it suffices to consider a sequence $(\zeta_m)_{m \geq 1}$ of functions such that, for all $m \geq 1$: $\zeta_m(\pi) = \chi$, for all π in $[0, \pi_m]$; ζ_m is continuous over $(\pi_m, 1)$; $\zeta_m(\pi_k) = \frac{1}{2}\left(\frac{\eta_k - \eta_{k+1}}{\pi_k - \pi_{k+1}} + \frac{\eta_{k-1} - \eta_k}{\pi_{k-1} - \pi_k}\right)$, for all $m > k > 1$; $\left|\zeta_m(\pi) - \frac{\eta_k - \eta_{k+1}}{\pi_k - \pi_{k+1}}\right| \leq \frac{1}{2}\left(\frac{2}{k} + \frac{2}{k+1}\right) = \frac{1}{k} + \frac{1}{k+1}$, for all π and $k < m$ such that $\pi \in [\pi_{k+1}, \pi_k]$; $\int_{\pi_{k+1}}^{\pi_k} (\zeta_m(\pi) - \sigma(\pi))\, d\pi = 0$, for all $m > k \geq 1$; ζ_m is odd, that is, $\zeta_m(-\pi) = \zeta_m(\pi)$, for all π; ζ_{m+1} is equal to ζ_m over $(\pi_{m+1}, 1)$. The sequence $(\zeta_m)_{m \geq 1}$ is then a Cauchy sequence for the norm of the uniform convergence. As it can be easily shown, its limit ζ is continuous and satisfies our requirements. A function $\widetilde{\eta}$ can then be simply defined as follows: $\widetilde{\eta}(\pi) = \eta_1 - \int_{\pi}^{\pi_1} \zeta(\pi)\, d\pi$. We have proved the lemma for $(\overline{\pi}, \overline{\eta}) = 0$.

In the general case, it suffices to obtain the function $\widetilde{\eta}$ for the sequence $(\pi_k - \overline{\pi}, \eta_k - \overline{\eta})_{k \geq 1}$ and to define the new function $\widetilde{\eta}(\pi - \overline{\pi}) + \overline{\eta}$. ||

Lemma A5-2: *Consider a system of differential equations $\frac{d}{dt} y(t) = h(t, y, \pi)$ and an initial condition $y(t_1) = \eta(\pi)$ that depend on a parameter π and that are defined over an open subset O of R^{n+2}, where n is the dimension of y. Assume that h is a continuous function from O to R^n such that $\frac{\partial}{\partial y_i} h$, $1 \leq i \leq n$, and $\frac{\partial}{\partial \pi} h$ exist and are continuous over O. Let $(\pi_k)_{k \geq 1}$ be a sequence in R such that $(t_1, \eta(\pi_k), \pi_k)_{k \geq 1}$ is a sequence in O that converges towards a point $(t_1, \overline{\eta}, \overline{\pi})$ in O. Assume also that $\lim_{k \to +\infty} \frac{\eta(\pi_k) - \overline{\eta}}{\pi_k - \overline{\pi}}$ exists and is finite. Let χ be this limit. Let $y(., \pi)$ be the solution of the differential system with the initial condition as a function of the parameter π.*

Then $\lim_{k \to +\infty} \frac{y(t, \pi_k) - y(t, \overline{\pi})}{\pi_k - \overline{\pi}}$ exists, for all t in the maximal definition interval of the solution $y(., \overline{\pi})$ and is equal to the solution ρ of the linear differential system $\frac{d}{dt} \rho(t) = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} h(t, y(t, \pi), \pi)\, \rho_i(t) + \frac{\partial}{\partial \pi} h(t, y(t, \pi), \pi)$ with initial condition $\rho(t_1) = \chi$.

Proof: The conclusion of the lemma will be proved if we prove it for all strictly monotonic subsequence of $(\pi_k)_{k\geq 1}$. We may thus assume that $(\pi_k)_{k\geq 1}$ is strictly monotonic. Through the change of variables $y = \eta(\pi) + z$, the initial system and initial condition are equivalent to $\frac{d}{dt}z(t) = h(t, \eta(\pi) + z, \pi)$ and $z(t_1) = 0$. From Lemma A5-1, there exists a continuously differentiable function $\widetilde{\eta}$ over a neighborhood of $\overline{\pi}$ that coincides with η over $\{\pi_k | k \geq 1\} \cup \{\overline{\pi}\}$. From the equality $\lim_{k\to+\infty} \frac{\eta(\pi_k)-\overline{\eta}}{\pi_k-\overline{\pi}} = \chi$, we have $\frac{d}{d\pi}\widetilde{\eta}(\overline{\pi}) = \chi$. The lemma then follows from the application of the standard theorems of the theory of ordinary differential equations about the differentiability of the solution with respect to a parameter to the system $\frac{d}{dt}z(t) = g(t, z, \pi)$, where $g(t, z, \pi) = h(t, \widetilde{\eta}(\pi) + z, \pi)$, with initial condition $z(t_1) = 0$. ||

Lemma A5-3: *Let f be a function from an open set O of R^n to R and let ω be an element of O. Assume that f is continuous at ω and that its partial derivatives $\frac{\partial}{\partial\tau_i}f(\omega)$, $1 \leq i \leq n$, exist. Assume also that $f \circ \tau$ is differentiable at 0 and that $\frac{d}{d\pi}f \circ \tau(0) = \sum_{i=1}^n \frac{\partial}{\partial\tau_i}f(\omega)\frac{d}{d\pi}\tau_i(0)$, for all continuously differentiable function $\tau(\pi)$ from $(-1, 1)$ to O such that $\tau(0) = \omega$. Then, f is differentiable at ω.*

Proof: Suppose that f is not differentiable at ω. Then, there exists $\epsilon > 0$ and a sequence $\left(\tau^k\right)_{k\geq 1}$ converging towards ω such that $\tau^k \neq \omega$, for all k, and

$$\left| \frac{f\left(\tau^k\right) - f(\omega)}{|\tau^k - \omega|} - \sum_{i=1}^n \frac{\partial}{\partial\tau_i}f(\omega)\frac{\left(\tau_i^k - \omega_i\right)}{|\tau^k - \omega|} \right| > \epsilon, \text{ (A5.1)}$$

for all $1 \leq k \leq n$. By extracting a subsequence, if necessary, we may assume that $\left(\left|\tau^k - \omega\right|\right)_{k\geq 1}$ is strictly decreasing. Since the sequence $\left(\frac{\tau^k-\omega}{\left|\tau^k-\omega\right|}\right)_{k\geq 1}$ is bounded, it admits a convergent subsequence. We may thus assume that this sequence itself is convergent. Let λ be its limit. Since every term of the sequence has a unit norm, this is also the case of the limit and we have $|\lambda| = 1$.

Applying Lemma A5-1 to $(\pi_k)_{k\geq 1} = \left(\left|\tau^k - \omega\right|\right)_{k\geq 1}$, $\overline{\pi} = 0$, $\left(\eta^k\right)_{k\geq 1} =$

$\left(\tau^k\right)_{k\geq 1}$, and $\overline{\eta} = \omega$, we obtain the existence of a continuously differentiable function $\widetilde{\tau}$ from $(-1,1)$ to R^n such that $\widetilde{\tau}(0) = \omega$ and $\widetilde{\tau}\left(\left|\tau^k - \omega\right|\right) = \tau^k$, for all $k \geq 1$ such that $\left|\tau^k - \omega\right| < 1$. Since $\lim_{k\to+\infty} \frac{\tau^k-\omega}{\left|\tau^k-\omega\right|} = \chi$, we have $\frac{d}{d\pi}\widetilde{\tau}(0) = \chi$. Then, from the assumptions of the lemma, $\frac{d}{d\pi} f \circ \tau(0)$ exists and is equal to $\sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\omega)\chi_i$. Consequently, the limit of the L.H.S. of (A5.1), for k tending towards infinity, exists and is equal to 0. This contradicts (A5.1) and the lemma is proved. ||

Lemma A5-4: *Let f be a function from an open set O in $R \times R^n$ to R and let $(\overline{u}, \omega)$ be an element of O. Assume that the function $f(\overline{u}, .)$ from $\{\tau \in R^n | (\overline{u}, \tau) \in O\}$ to R is differentiable at ω and that $\frac{\partial}{\partial u} f$ exists in O and is continuous at $(\overline{u}, \omega)$. Then, f is differentiable at $(\overline{u}, \omega)$.*

Proof: We will have proved the lemma if we prove that the limit of the ratio below for (u, τ) tending towards $(\overline{u}, \omega)$ exists and is equal to 0:

$$\frac{\left| f(u,\tau) - f(\overline{u},\omega) - \frac{\partial}{\partial u} f(\overline{u},\omega)(u - \overline{u}) - \sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\overline{u},\omega)(\tau_i - \omega_i)\right|}{\left|(u - \overline{u}, \tau - \omega)\right|}.$$

However, this ratio is not larger than (A5.2) below:

$$\left|\frac{f(u,\tau) - f(\overline{u},\tau)}{u - \overline{u}} - \frac{\partial}{\partial u} f(\overline{u},\omega)\right| \frac{|u - \overline{u}|}{|(u - \overline{u}, \tau - \omega)|} + \frac{\left| f(\overline{u},\tau) - \sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\overline{u},\omega)(\tau_i - \omega_i)\right|}{|\tau - \omega|} \frac{|\tau - \omega|}{|(u - \overline{u}, \tau - \omega)|}. \quad \text{(A5.2)}$$

Obviously, the two factors $|u - \overline{u}| / |(u - \overline{u}, \tau - \omega)|$ and $|\tau - \omega| / |(u - \overline{u}, \tau - \omega)|$ are not larger 1. From the mean value theorem, $\frac{f(u,\tau) - f(\overline{u},\tau)}{u - \overline{u}} = \frac{\partial}{\partial u} f(u', \omega)$, where u' lies strictly between u and $\overline{u}$. As (u, τ) tends towards $(\overline{u}, \omega)$, (u', τ) also tends towards $(\overline{u}, \omega)$ and, from the continuity of $\frac{\partial}{\partial u} f$ at $(\overline{u}, \omega)$, $\frac{\partial}{\partial u} f(u', \tau)$ tends towards $\frac{\partial}{\partial u} f(\overline{u}, \omega)$. Consequently, the first term in (A5.2) tends towards 0. From the differentiability (with respect to τ) of $f(\overline{u}, .)$ at ω, the second term also tends towards 0 and the lemma is proved. ||

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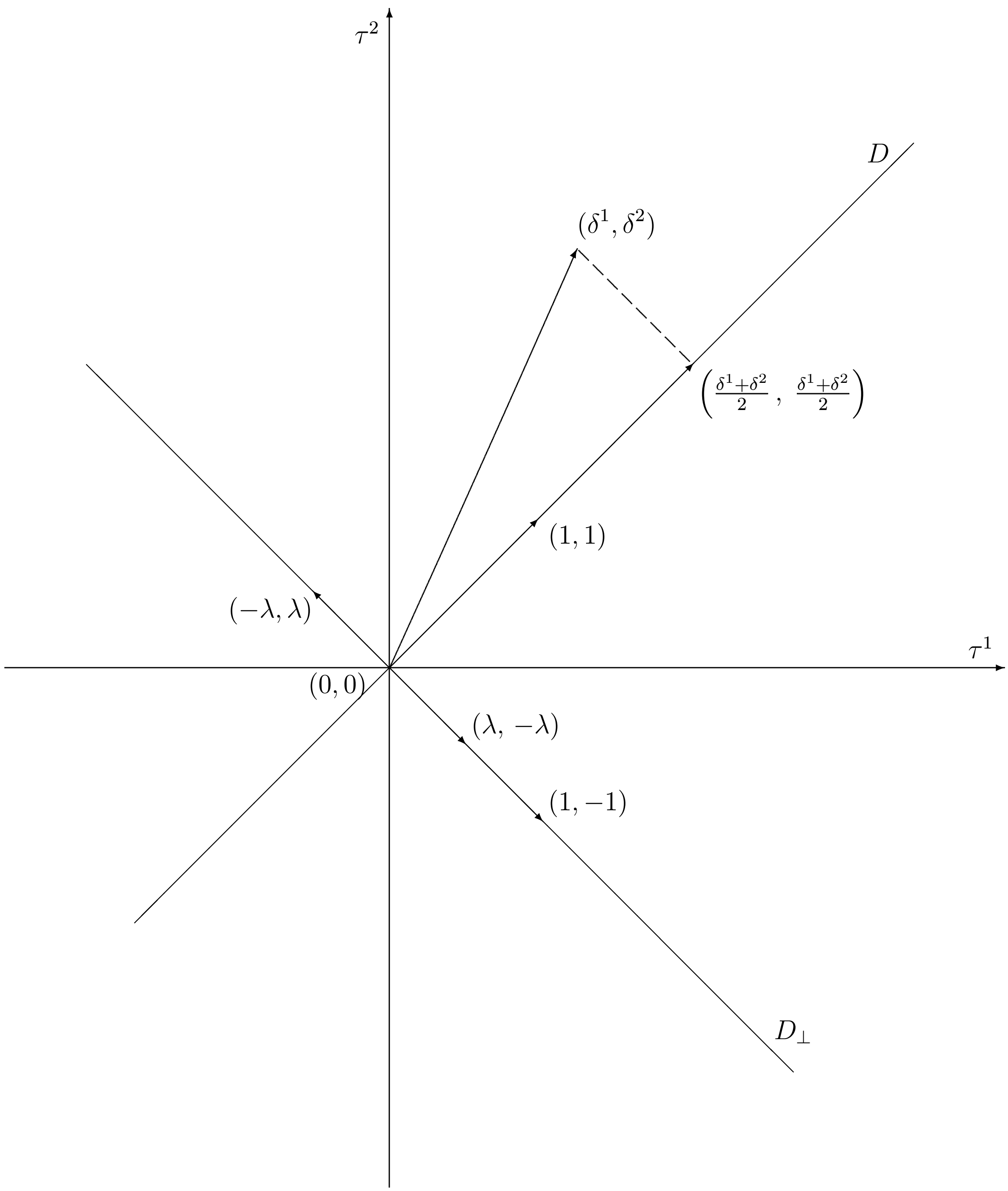

τ^2
D
(δ^1, δ^2)
$\left(\frac{\delta^1+\delta^2}{2}\ ,\ \frac{\delta^1+\delta^2}{2}\right)$
$(1, 1)$
$(-\lambda, \lambda)$
τ^1
$(0, 0)$
$(\lambda, -\lambda)$
$(1, -1)$
$D_\perp$

Figure 1