First-Price and Second-Price Auctions with Resale

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Abstract

We add a resale stage to standard auctions with two bidders. Bids are either kept secret or made public. Either the auction winner or the auction loser chooses the resale price. We characterize an infinity of equilibria of the second-price auction and a unique equilibrium of the first-price auction. For every equilibrium of an auction without bid disclosure, we construct an outcome-equivalent and, in the case of the second-price auction, "posterior implementable" equilibrium of the auction with bid disclosure. We compare the revenues from the two auctions and from the two bargaining procedures at resale.

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1. Introduction

Most theoretical models of auction forbid resale between bidders. However, resale is, at least, possible after many real-life auctions. Documented examples include auctions of bonds, bills, foreign exchange, timber rights, SO_2 emission allowances, radio-wave spectrum licences, and gold.

We consider the standard independent private value model with two, possibly heterogeneous, bidders. To the first- and second-price auctions, we add a stage where resale between bidders may occur. At resale, either the auction winner or the auction loser makes a take-it-or-leave-it offer. The auctions are either sealed, in which case bids are kept secret before resale and payments in the second-price auction are deferred until after resale, or open, in which case bids are made public before resale. Obviously, since the payment is often revealed immediately after or during the auction, assuming only with secret bids would be inadequate to many applications. Our results apply as well to intermediate assumptions on the release of bids; for example, to the Dutch auction, where only the higher bid is revealed, and to the English auction, where only the lower bid is revealed.

We link these auctions with resale to auctions with (pure) common value. In any equilibrium, the bidders' net values for winning the auction are identical when their bids are identical. From this link, we obtain and explicitly characterize an infinity of equilibria of the second-price auction and a unique equilibrium of the first-price auction. In general, these equilibria are pure for sealed-bid auctions and "mixed," or, more correctly, "behavioral," for open auctions. Surprisingly enough, for every equilibrium of a sealed-bid auction, there exists an "outcome-equivalent" equilibrium of the open auction.

The usual "truth-bidding" equilibrium of the second-price auction without resale remains an equilibrium if resale is allowed. However, resale destroys its weak-dominance (as Gupta and Lebrun 1999 noticed)[1] and allows an infinity of (inefficient) equilibria. All our equilibria of the open second-price auction are "posterior implementable," that is, no bidder would regret his bid upon learning the other bidder's bid.

In the case with heterogenous bidders, we prove, by examining the final equilibrium allocations, that some equilibria of the second-price auction bring more revenues to the auctioneer that the equilibrium of the first-price auction and that some others bring less.

Although the equilibrium bid distributions of the first-price auctions with different bargaining procedures cannot be stochastically ranked, we show, again by focusing on the final allocations, that the auctioneer's revenues are higher when the price setter at resale is the auction winner.

Contrary to Gupta and Lebrun (1999), where all private information is exogenously released after the auction, only endogenous release of information, through the auction outcome and bids, occurs in our model. Assuming resale under complete information as in Gupta and Lebrun (1999) would make the link with the common-value model immediate. Resale would remedy any inefficiency and bidders' net values for winning the auction would be equal to the resale price. In the present paper, information is incomplete at resale and, as is expected from Myerson and Satterwhaite (1983), not all inefficiencies are remedied after the auction. Nevertheless, in asymmetric equilibria, resale does take place and bidders' net values coincide when bids are identical. We show that this commonality of net values for identical or "pivotal" bids is enough to extend standard results from the common-value model.

Tröger (2003) and Garratt and Tröger (2003, 2005) add to the standard symmetric model one "speculator"–a bidder whose only interest is in reselling the item. In our paper, both bidders have also a "use value" for the item. Garatt and Tröger do not consider the link with common-value model, nor

[1] It is the only ex-post equilibrium, see Krishna and Hafalir (2006).

general comparisons between equilibrium payoffs.

In a similar augmented symmetric model, Bose and Deltas (2004) show the presence of a winner's curse in auctions between speculators and one final consumer, even when, as they assume, the speculators are not allowed to act on the information they gather at auction. In our paper, the presence of a winner's curse is made obvious through our link with the common-value model.

When bidders observe only noisy signals of their use values before the auction, Haile (2000) assumes that information becomes complete before resale. He obtains a model with affiliated exogenous net values, to which he can apply Milgrom and Weber (1982)'s methods. In a similar model with private uncertainty, Haile (2003) also addresses resale under incomplete information and obtains formulas for the pure symmetric separating equilibrium, conditional on its existence.

Using calculus of variation, Krishna and Hafalir (2006) show that, no matter the bargaining procedure at resale[2], the unique equilibrium of the first-price auction brings more revenues to the auctioneer than the truth-bidding equilibrium of the second-price auction. Krishna and Hafalir (2006) examine only the secret-bid case and do not offer comparative statics results pertaining to a change of the bargaining procedure. They exhibit, in a few examples, equilibria different from the truth-bidding equilibrium, but do not offer a general characterization.

Milgrom (1987) considers resale in auctions with complete information throughout. Pagnozzi (2005) shows that resale may occur at the equilibrium of a second-price auction that awards a project with random cost to one of two heterogenous bidders, one with limited liability and none with private information.

Optimal mechanism design under the presence of resale has been studied

[2] In addition to our two ultimatums procedures, Krishna and Hafalir (2006) also consider "intermediate" procedures where the price setter at resale is chosen at random according to exogenous probabilities.

in Zheng (2002), Calzolari and Pavan (2003), and Lebrun (2005).

2. Sealed-Bid Auctions

Bidder 1 and bidder 2's use values for the item being auctioned are independently distributed over the same interval $[c, d]$, with $c < d$, according to absolutely continuous probability measures F_1 and F_2 with density functions f_1 and f_2 that are strictly positive and continuous[3]. We use the same notations F_1 and F_2 for the cumulative distribution functions. A bidder's use value is his own private information.

For the sake of simplicity, we assume that there is no reserve price and that participation to the auction is mandatory[4]. Bids are not revealed before resale. If, as it is natural, the winner of the second-price auction learns the price when he pays it, payment to the auctioneer has to be deferred until after the resale stage.

Resale takes place at the resale stage if and only if the price setter proposes a resale price the other bidder agrees to. We first define a regular equilibrium.

Definition 1:

(i) A regular bidding function β_i *of bidder* i *is a strictly increasing and continuous function from* $[c, d]$ *to* $[c, +\infty)$.

(ii) If the auction winner (loser) is the price-setter at resale, a regular resale-offer function γ_i *of bidder* i *is a real-valued, bounded, and measurable*[5] *function defined over* $[c, d] \times [c, +\infty)$ *and such that* $\gamma_i(v; b) \geq (\leq) v$, *for all* $(v; b)$ *in* $[c, d] \times [c, +\infty)$.

[3]Many of our results hold true under more general assumptions that allow, for example, density functions that are defined and strictly positive only over $(c, d]$ (as long as they are bounded).

[4]Our equilibria remain equilibria if participation is voluntary. Our results about the second-price auction easily extend to the case with an arbitrary reserve price.

[5]As everywhere in this paper, with respect to the σ-algebras of Borel subsets.

(iii) A regular strategy of bidder i is a couple $\sigma_i = (\beta_i, \gamma_i)$ where β_i is a regular bidding function and γ_i a regular resale-offer function.

(iv) A regular equilibrium $(\sigma_1, \sigma_2) = (\beta_1, \gamma_1; \beta_2, \gamma_2)$ is a couple of regular strategies that can be completed[6] *into a perfect Bayesian equilibrium.*

If bidder i with use value v_i follows (β_i, γ_i), he bids $\beta_i(v_i)$ at auction and offers the resale price $\gamma_i(v; b_i)$ at resale when he is the price setter and when he has submitted b_i at auction.

We then define the optimal-resale-price functions ρ^s and ρ^b as follows.

Definition 2:

(i) For all $i = 1, 2$, the buyer's virtual-use-value function ω_i^b and the seller's virtual-use-value function ω_i^s are defined over $[c, d]$ as follows:

$$\omega_i^b(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}; \omega_i^s(v_i) = v_i + \frac{F_i(v_i)}{f_i(v_i)}.$$

(ii) Let ω_i^b (ω_i^s) be strictly increasing over $[c, d]$, for $i = 1, 2$. Let the seller (buyer)'s optimal-resale-price function ρ^s (ρ^b) be the function defined over $[c, d]^2$ such that, for all (w_1, w_2) in $[c, d]^2$, its value at (w_1, w_2) is equal to the unique solution of the equation below:

$$w_k = \rho^x(w_1, w_2) - \frac{F_l(w_l) - F_l(\rho^x(w_1, w_2))}{f_l(\rho^x(w_1, w_2))}, \quad (1)$$

where $x = s$ (b) and k and l are such that $l \neq k$ and $w_k \leq (\geq)\ w_l$.

(iii) Notation:

$\rho_1^x(v, w) = \rho_2^x(w, v) = \rho^x(v, w)$, *for all (v, w) in $[c_1, d_1] \times [c_2, d_2]$ and $x = s, b$.*

In Definition 2 (ii), $\rho^s(w_1, w_2)$ ($\rho^b(w_1, w_2)$) is the resale price that maximizes bidder k's expected payoff when bidder k's use value is w_k and bidder

[6] By adding beliefs and by adding what responses every bidder should give to offers from the other bidder at resale, as functions of the past observed histories.

l's use value is distributed according to F_l conditionally on belonging to the interval $[c, w_l]$ ($[w_l, d]$). That equation (1) has a unique solution follows easily from the strict monotonicity of ω_i^y, $y \neq x$. According to the notation (iii), ρ_i^x is the function ρ^x where we write bidder i's use value as the first argument.

In Theorems 1 and 2 below, as everywhere in this paper, β_i^{-1} denotes the "extended" inverse of β_i that takes the constant value d above the range of β_i.

Theorem 1: *Let x, y be s or b and such that $x \neq y$. Let ω_1^y and ω_2^y be strictly increasing. Let φ be a strictly increasing continuous function over $[c, d]$ such that $\varphi(c) = c$ and $\varphi(d) = d$. Let $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ be the following couple of regular strategies:*

$$\begin{aligned} \beta_1(v) &= \rho^x(v, \varphi(v)), \\ \beta_2(v) &= \rho^x\left(\varphi^{-1}(v), v\right), \end{aligned}$$

for all v in $[c, d]$;

$$\begin{aligned} \gamma_i(v; b) &= \rho_i^s(v_i, \max(v_i, \alpha_j(b))), \text{ if } x = s, \\ \gamma_i(v; b) &= \rho_i^b(v_i, \min(v_i, \alpha_j(b))), \text{ if } x = b, \end{aligned}$$

where $\alpha_i = \beta_i^{-1}$, $i = 1, 2$, for all (v, b) in $[c, d] \times [c, +\infty)$. Then, $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ is a regular equilibrium of the second-price auction where payments are deferred and bids are kept secret and where the price setter at resale is the auction winner if $x = s$ and the auction loser if $x = b$. Moreover, the following equalities hold true:

$$\begin{aligned} \alpha_2 \beta_1 &= \varphi, (2) \\ \rho^x(\alpha_1(b), \alpha_2(b)) &= b, \end{aligned}$$

for all v *in* $[c,d]$ *and* b *in* $[\beta_1(c), \beta_1(d)] = [\beta_2(c), \beta_2(d)] = [c,d]$.

Since there is an infinity of functions φ as in Theorem 1, the sealed-bid second-price auction with resale has, like the common-value auction, an infinity of equilibria. From (2) above, φ determines the "intermediate" equilibrium allocation, that is, the allocation after the auction and before resale.

If φ is the identity function, every bidder submits at equilibrium his use value. Since the auction efficiently allocates the item between bidders, no resale takes place in this equilibrium. It is outcome-equivalent to the equilibrium in weakly dominant strategies of the second-price auction with no resale allowed. Of course, this equivalence no longer holds true for equilibria constructed from functions φ different from the identity function.

Theorem 2: *Let* x, y *be* s *or* b *and such that* $x \neq y$. *Let* ω_1^y *and* ω_2^y *be strictly increasing. Let* $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ *be the following couple of regular strategies:*

$$\beta_i(v) = \frac{\int_0^{F_i(v)} \rho^x \left(F_1^{-1}(q), F_2^{-1}(q)\right) dq}{F_i(v)} \quad (3)$$

for all v *in* $[c,d]$*;*

$$\begin{aligned} \gamma_i(v;b) &= \rho_i^s(v_i, \max(v_i, \alpha_j(b))), \text{ if } x = s, \\ \gamma_i(v;b) &= \rho_i^b(v_i, \min(v_i, \alpha_j(b))), \text{ if } x = b, \end{aligned}$$

where $\alpha_i = \beta_i^{-1}$, $i = 1, 2$, *for all* (v,b) *in* $[c,d] \times [c, +\infty)$. *Then,* $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ *is a regular equilibrium of the first-price auction where bids are kept secret and where the price setter at resale is the auction winner if* $x = s$ *and the auction loser if* $x = b$. *The following equality holds true:*

$$\alpha_2 \beta_1 = F_2^{-1} F_1. (4)$$

Moreover, β_1, β_2 are the unique bidding functions that are differentiable over $(c, d]$ and part of a regular equilibrium.

From Gupta and Lebrun (1999), the equilibrium bidding functions (3) are the same as in the simple model where private information becomes public before resale and the resale price is exogenously determined according to ρ^x. As (4) indicates, the bids are distributed identically across bidders. Indeed, from Gupta and Lebrun (1999), the same bid distributions arise at the equilibrium of the symmetric model where both bidders' use values are distributed according to G^x such that $(G^x)^{-1}(q) = \rho^x\left(F_1^{-1}(q), F_2^{-1}(q)\right)$, for all q in $[0, 1]$.

The intuition for the theorems above and the main argument of their proofs come from a link, we now describe, between our model and the common-value model. Assume that β_1, β_2 are the bidding strategies the bidders are expected to follow at auction. Bidder i's updated beliefs about bidder j's use value after winning (losing) the auction with a bid b_i are represented by the conditional of F_j on $[c, \alpha_j(b_i)]$ $([\alpha_j(b_i), d])$. Then, $\gamma_i(v_i; b_i) = \rho_i^s(v_i, \max(v_i, \alpha_j(b_i)))$ $(\rho_i^b(v_i, \min(v_i, \alpha_j(b))))$, as in Theorem 1, is the smallest (largest) resale price that maximizes his expected payoff. Assume that the bidders choose their resale prices according to these resale-offer functions. Then, the bidders' net values for winning will be as in Lemma 1 below.

Lemma 1: *Let x, y be s or b and such that $x \neq y$. Let ω_1^y and ω_2^y be strictly increasing. Assume bidder i expects bidder j to follow a regular bidding function β_j, for all $i \neq j$. Assume bidder i offers the resale price according to the optimal regular resale-offer function γ_i such that $\gamma_i(v; b) = \rho_i^s(v, \max(v, \alpha_j(b)))$ $(\rho_i^b(v, \min(v, \alpha_j(b))))$ if $x = s$ (b), for all $i = 1, 2$ and (v, b) in $[c, d] \times [c, +\infty)$. Then, bidder i's net-value function u_i^x, that is, the difference between his utility $u_i^{x,w}$ when winning (gross of the* *auction price) and his utility* $u_i^{x,l}$ *when losing is as follows:*

$$
\begin{aligned}
& u_i^s\left(v_i, v_j; b_i, b_j; \beta_i, \beta_j\right) \\
= \; & \rho_i^s\left(v_i, \max\left(v_i, \alpha_j\left(b_i\right)\right)\right), \text{ if not larger than } v_j; \\
= \; & \rho_j^s\left(v_j, \max\left(v_j, \alpha_i\left(b_j\right)\right)\right), \text{ if not larger than } v_i; \\
= \; & v_i, \text{ otherwise;}
\end{aligned}
$$

$$
\begin{aligned}
& u_i^b\left(v_i, v_j; b_i, b_j; \beta_i, \beta_j\right) \\
= \; & \rho_i^b\left(v_i, \min\left(v_i, \alpha_j\left(b_i\right)\right)\right), \text{ if not smaller than } v_j; \\
= \; & \rho_j^b\left(v_j, \min\left(v_j, \alpha_i\left(b_j\right)\right)\right), \text{ if not smaller than } v_i; \\
= \; & v_i, \text{ otherwise;}
\end{aligned}
$$

for all couple of bids (b_1, b_2) *in* $[c, +\infty)^2$*, couple of use values* (v_1, v_2) *in* $[c, d]^2$*, and* $i, j = 1, 2$ *with* $i \neq j$*.*

When resale could take place at the price one of the two bidders would offer, bidder i's net value is equal to the resale price: by winning bidder i saves the resale price if he would be a buyer at resale and earns it if he would be a reseller. Otherwise, bidder i's net value is, as when resale is forbidden, equal to his use value: winning secures him the item, which he could not have obtained at resale. Since it depends, through their inverses, on the bidding functions the bidders are expected to follow, the net value is "endogenous."

When looking for regular equilibria in the case $x = s\ (b)$, we may, as we do below, focus on bidder i's expected utility up to his expected utility from losing (winning) with probability one. In fact, since bidder j, $j \neq i$, does not observe b_i when he makes an offer at resale, bidder i's utility $u_i^{s,l}$ $(u_i^{b.w})$ when losing (winning) does not depend on his bid b_i.

Since bidder i's bid b_i can enter his net value only as an argument of his resale price, which, we have assumed, he chooses optimally, $b_i' = b_i$ is a solution of the maximization problem below:

$$b_i \in \arg\max_{b_i' \geq c} \int_e^{\alpha_j(b_i)} u_i^x \left(v_i, v_j; b_i', \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j), \quad (5)$$

where $e = c$ if $x = s$ and $e = d$ if $s = b$. By applying the integral form of (a variant of) the envelope theorem in Milgrom and Segal (2002) to this problem, we prove in Appendix 1 Lemma 2 (i) below, which allows to circumvent the direct dependence of u_i^x on the own bid b_i. We also prove in Appendix 1 the rest of Lemma 2.

Lemma 2: *Let x, y be s or b and such that $x \neq y$. Let ω_1^y and ω_2^y be strictly increasing. Assume bidder i expects bidder j to follow the regular bidding function β_j, for all $i \neq j$. For all $i = 1, 2$, let u_i^x be bidder i's net-value function as defined in Lemma 1. Then, for all $i \neq j$:*

(i)–Envelope Result: For all (v_i, b_i) in $[c, d] \times [c, +\infty)$,

$$\int_e^{\alpha_j(b_i)} u_i^x \left(v_i, v_j; b_i, \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j)$$
$$= \int_e^{\alpha_j(b_i)} u_i^x \left(v_i, v_j; \beta_j(v_j), \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j),$$

where $e = c$ if $x = s$ and $e = d$ if $x = b$.

(ii)–Common Value for Identical Bids when Bidding as Expected:

For all $b \geq c$,

$$u_1^x(\alpha_1(b), \alpha_2(b); b, b; \beta_1, \beta_2) = u_2^x(\alpha_1(b), \alpha_2(b); b, b; \beta_1, \beta_2) = \rho^x(\alpha_1(b), \alpha_2(b)).$$

(iii)–Monotonicity with respect to Own Type: For all $b \geq c$, $u_i^x \left(v_i, \alpha_j(b); b, b; \beta_i, \beta_j\right)$ is nondecreasing with respect to v_i in $[c, d]$.

Optimal resale under incomplete information at least remedies the "worst cases" of inefficiency, where, given a price setter's use value, the other bidder's use value is as far away as possible, that is, when both bidders submit the same bid. Then, as stated in Lemma 2 (ii), the equality between both bidders' net values and the resale price, which always holds true under complete information (as in Gupta and Lebrun, 1999), also holds true in those cases.

Theorems 1 and 2 follow easily from Lemma 2. Indeed, from Lemma 2 (i), bidder i's expected net payoffs when his use value is v_i and his bid is b are as follows.

Expected Net Payoff in the Second-Price Auction:

$$\int_e^{\alpha_j(b)} u_i^x \left(v_i, v_j; \beta_j(v_j), \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j) - \int_c^{\alpha_j(b)} \beta_j(v_j)\, dF_j(v_j); \ (6)$$

Expected Net Payoff in the First-Price Auction:

$$\int_e^{\alpha_j(b)} u_i^x \left(v_i, v_j; \beta_j(v_j), \beta_j(v_j); \beta_i, \beta_j\right) dF_j(v_j) - \int_c^{\alpha_j(b)} b dF_j(v_j); \ (7)$$

where $e = c$ if $x = s$ and $e = d$ if $x = b$.

Since, at an equilibrium, b should be optimal if $v_i = \alpha_i(b)$, we obtain from Lemma 2 (ii) the following first-order conditions.

First-Order Equilibrium Condition in the Second-Price Auction:

$$\rho^x(\alpha_1(b), \alpha_2(b)) = b. \ (8)$$

First-Order Equilibrium Conditions in the First-Price Auction (if the bidding functions are differentiable):

$$\frac{d}{db} \ln F_i(\alpha_i(b)) = \frac{1}{\rho^x(\alpha_1(b), \alpha_2(b)) - b}, \ i = 1, 2. \ (9)$$

The same first-order conditions (9) would follow from any other choice of optimal regular resale-offer functions. From Gupta and Lebrun (1999), the bidding functions described in Theorem 2 form the unique solution of the two conditions (9)[7]. As in the common-value second-price auction, the multiplicity of equilibria described in Theorem 1 ensues from the single condition (8). From Lemma 2 (iii), the "second-order" condition is satisfied for both auction procedures and the expected net payoff, which is then quasi-concave with respect to the bid, reaches its maximum at the bid the bidding function specifies.

Corollary 1 below describes a property of the equilibria we will use in the following section to construct equivalent behavioral equilibria of the open auctions.

Corollary 1: *Let $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ be a regular equilibrium as in Theorem 1 or Theorem 2 and let φ be equal to $\alpha_2\beta_1$. Let φ^+ and φ^- be the functions defined over $[c, d]$ as follows:*

$$\begin{aligned} \varphi^+(v) &= \min\{w \in [v, d] \,|\varphi(w) = w\} \\ \varphi^-(v) &= \max\{w \in [c, v] \,|\varphi(w) = w\}. \end{aligned}$$

If the price setter at resale is the auction winner, then all bids in

$$\left[\beta_1(\varphi^-(v)), \rho^s(\alpha_1(.), \alpha_2(.))^{-1}(v)\right]$$

are optimal for bidder 1 (2) with use value v in $[c, d]$ such that $\varphi(v) <(>) v$. If the price setter at resale is the auction loser, then all bids in

$$\left[\rho^b(\alpha_1(.), \alpha_2(.))^{-1}(v), \beta_1(\varphi^+(v))\right]$$

are optimal for bidder 1 (2) with use value v in $[c, d]$ such that $\varphi(v) >(<) v$.

For example, assume the price setter at resale is the auction winner and

[7] Together with the immediate boundary condition $\beta_1(d) = \beta_2(d)$.

$\varphi(v) > v$, that is, $\beta_1(v) > \beta_2(v)$ (the proof is similar in the other cases). Since bidder 2 with use value v wins the auction only if bidder 1's use value is smaller v, no trade could occur with bidder 2 as the price setter. Let b be a bid in $\left(\beta_1(\varphi^-(v)), \rho^s(\alpha_1(.), \alpha_2(.))^{-1}(v)\right)$, where $\varphi^-(v)$ is the largest point of coincidence between the bidding functions to the left of v. Then, $\alpha_1(b) < \alpha_2(b)$ and $\rho^s(\alpha_1(b), \alpha_2(b)) < \min(v, \alpha_2(b))$.

By continuity, there exists a neighborhood of $\alpha_1(b)$ such that, for all v_1 in this neighborhood, $\rho^s(v_1, \alpha_2(\beta_1(v_1)))$ is smaller than v and $\alpha_2(b)$ and is thus, from Lemma 1, equal to $u_2^s(v_1, v; \beta_1(v_1), \beta_1(v_1); \beta_1, \beta_2)$ and $u_1^s(v_1, \alpha_2(b); \beta_1(v_1), \beta_1(v_1); \beta_1, \beta_2)$, both. Consequently, from (6) and (7), the first-order effect of a bid change from b on bidder 2's expected payoff when his use value is v is the same as when his use value is $\alpha_2(b)$. Since, by the equilibrium condition, this first-order effect vanishes, bidder 2's expected payoff must be constant over the closure of $\left(\beta_1(\varphi^-(v)), \rho^s(\alpha_1(.), \alpha_2(.))^{-1}(v)\right)$. The equilibrium bid $\beta_2(v)$ belongs to this interval and, consequently, all bids in it are optimal.

Intuitively, when bidder 2 is not the price setter and would accept the resale offer from the other bidder, the first-order effect of a bid change is determined by the resale price and hence is independent on bidder 2's use value. Since the bid is optimal for a certain use value, the first-order effect vanishes throughout.

For the second-price auction, the set of optimal bids in the example above is simply, from Theorem 1, $[\varphi^-(v), v]$.

From Theorem 2, the function φ in Corollary 1 is $F_2^{-1}F_1$ for the first-price auction.

Corollary 2 below characterizes the final, that is, after resale, equilibrium allocations.

Corollary 2: *Let $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ be a regular equilibrium as in Theorem 1 or Theorem 2 and let φ be equal to $\alpha_2\beta_1$. Let x be s (b) if the price setter at resale is the auction winner (loser). Let λ_φ^x be the function defined over*

$[c,d]$ *as follows:*

$$\lambda^s_\varphi(v_1) = \rho^s(v_1,\varphi(v_1)), \text{ if } \varphi(v_1) \geq v_1;$$
$$\lambda^s_\varphi(v_1) = \rho^s\left(\varphi^{-1}(.),.\right)^{-1}(v_1), \text{ if } \varphi(v_1) \leq v_1;$$

$$\lambda^b_\varphi(v_1) = \rho^b(v_1,\varphi(v_1)), \text{ if } \varphi(v_1) \leq v_1;$$
$$\lambda^b_\varphi(v_1) = \rho^b\left(\varphi^{-1}(.),.\right)^{-1}(v_1), \text{ if } \varphi(v_1) \geq v_1.$$

If bidder 1's and bidder 2's use values v_1, v_2 in $[c,d]$ are such that $v_2 <(>)\lambda^x_\varphi(v_1)$, then the equilibrium eventually allocates the item to bidder 1 (2).

The proof is simple. For example, assume $x = s$ and $\varphi(v_1) \geq v_1$. From the definitions of φ and λ^s_φ, we have $\lambda^s_\varphi(v_1) \leq \varphi(v_1) = \alpha_2\beta_1(v_1)$. If $v_2 < \lambda^s_\varphi(v_1)$, bidder 2 loses the auction and refuses bidder 1's resale offer. If $\lambda^s_\varphi(v_1) < v_2 < \varphi(v_1)$, bidder 2 loses the auction and accepts bidder 1's resale offer. If $\varphi(v_1) < v_2$, bidder 2 wins the auction and no advantageous resale is possible.

3. Open Auctions

We now turn to the auctions where bids are publicly revealed prior to resale. We need to extend our definition of a regular equilibrium by allowing behavioral strategies.

Definition 3:

(i) A regular bidding strategy $G_i(.|.)$ is a regular conditional probability measure with respect to v_i in $[c,d]$.

(ii) A regular strategy is a couple $(G_i(.|.),\delta_i)$ where $G_i(.|.)$ is a regular bidding strategy and δ_i is a regular resale-offer function, as defined in Definition 1 (ii).

(iii) Bidder i's regular beliefs are represented by a regular conditional probability measure $F_j(.|.)$ *with respect to* b_j *in* $[c, +\infty)$.

(iv) A regular equilibrium is a couple of regular strategies and a couple of regular beliefs $(G_1(.|.), \delta_1, F_2(.|.); G_2(.|.), \delta_2, F_1(.|.))$ *that can be completed into a perfect Bayesian equilibrium.*

If bidder i with use value v_i follows $(G_i(.|.), \delta_i)$, he chooses his bid according to $G_i(.|v_i)$ and, if he is the price setter and bidder j bids b_j, he offers $\delta_i(v_i; b_j)$ at resale. Here, contrary to the previous section, the second argument of δ_i is bidder j's bid. The measure $F_j(.|b_j)$ represents the revised beliefs bidder i holds about bidder j's use value after having observed bidder j's bid b_j. We have Corollary 3 below.

Corollary 3: *Let* $\mathcal{E}$ *be a regular equilibrium of a sealed-bid auction as in Theorem 1 or Theorem 2 (Section 2). Then, there exists a regular equilibrium* $\mathcal{E}'$ *of the open auction such that:*

(i) The bid marginal distributions, the interim total expected payoffs, and the final allocation are the same as in $\mathcal{E}$*;*

(ii) Conditionally on the use value of the price setter at resale, resale takes place with the same probability as in $\mathcal{E}$ *and, when this probability is different from zero, at the same price;*

(iii) If the auction is the second-price auction, the auction outcomes–the bids and the allocation before resale–are posterior implemented by $\mathcal{E}'$.

From (i) and (ii), for every regular equilibrium of a sealed-bid auction, there exists an "equivalent" regular equilibrium of the open auction. Following Green and Laffont (1987) (see, also, Lopomo 2001), (iii) means that all bids in the support of bidder i's bidding strategy conditional on v_i are optimal for bidder i with use value v_i even after he learns bidder j's bid.

Let $\mathcal{E} = (\beta_1, \gamma_1; \beta_2, \gamma_2)$ be a regular equilibrium as in Theorem 1 or Theorem 2. We prove Corollary 3 by constructing an equilibrium $\mathcal{E}' =$

$(G_1(.|.), \delta_1, F_2(.|.); G_2(.|.), \delta_2, F_1(.|.))$ with the required properties. The construction proceeds into the following four steps.

Step 1. Construction of the supports: If, conditionally on his use value v_i, no resale could occur with bidder i as the price setter, the support of $G_i(.|v_i)$ is the interval of optimal bids described in Corollary 1 (Section 2). Otherwise, the support of $G_i(.|v_i)$ is $\{\beta_i(v_i)\}$.

Step 2. Construction of revised beliefs $F_1(.|.)$ and $F_2(.|.)$ that are consistent with the supports in Step 1 and such that, when advantageous resale is possible, the price setter finds it optimal to offer the same resale price he offers in $\mathcal{E}$.

Step 3. Construction of the bidding strategy $G_i(.|.)$ as the conditional distribution of the bid with respect to the use value from the joint distribution of the use-value-bid couples generated by the marginal $F_i\alpha_i$ of bidder i's bid in $\mathcal{E}$ and the conditional $F_i(.|.)$ from Step 2, for all $i = 1, 2$.

Step 4. Extension of the construction of optimal regular resale-offer functions from the domains in Step 2, where resale is possible, to the whole definition domain $[c, d] \times [c, +\infty)$ such that, when the price setter is the auction winner (loser), the resale offer does not depend on the bid from the auction loser (winner) along the equilibrium path.

Step 1 is a simple definition. To show that Step 2 can be carried out, assume $x = s$ and there exists v such that $\varphi(v) > v$ and consider b_2 in $(\beta_2(\varphi^-(v)), \beta_2(\varphi^+(v)))$ (see Figure 1). From Step 1 and Corollary 1 (Section 2), the support of $F_2(.|b_2)$ must be $I = [\rho^s(\alpha_1(b_2), \alpha_2(b_2)), \varphi^+(v)]$. In order for bidder 1 with use value $v_1 = \alpha_1\left(\rho^s(\alpha_1(.), \alpha_2(.))^{-1}(w)\right)$, with w in I, to propose the same resale price w as in $\mathcal{E}$, it must maximize $(w - v_1)(1 - F_2(w|b_2))$ and hence satisfy (assuming differentiability) the first-order condition below:

$$\frac{d}{dw}\ln(1 - F_2(w|b_2)) = \frac{1}{\alpha_1\left(\rho^s(\alpha_1(.), \alpha_2(.))^{-1}(w)\right) - w}. \quad (10)$$

If this necessary first-order condition is satisfied for all w in the interior of I, it will also be sufficient (since $w+\frac{1-F_2(w|b_2)}{f_2(w|b_2)}$ will then be equal to the increasing function $\alpha_1\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)$. Integrating this equation in w from the lower extremity of I to v_2 in the interior of I, we find:

$$F_2\left(v_2|b_2\right)=1-\exp\int_{\rho^s(\alpha_1(b_2),\alpha_2(b_2))}^{v_2}\frac{1}{\alpha_1\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)-w}dw. \quad (11)$$

This equation defines indeed a probability distribution over I since the limit of its right-hand side tends towards one as v_2 tends towards $\varphi^+\left(v\right)$ (for the proof, see Appendix 2).

Step 3 leads to a bidding strategy of bidder i if and only if the marginal distribution F_i^* of the joint distribution generated by $F_i\left(.|.\right)$ and $F_i\alpha_i$ is equal to the actual distribution F_i of bidder i's use value. To show that this is indeed the case, assume, as in the previous paragraph, that $x=s$ and there exists v such that $\varphi\left(v\right)>v$ (see Figure 1). From Step 2, (10) or, equivalently, (12) below holds true, for all w in $\left(\varphi^-\left(v\right),\varphi^+\left(v\right)\right)$ and all b_2 in $\left[\beta_2\left(\varphi^-\left(v\right)\right),\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right]$:

$$\left(w-\alpha_1\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)\right)f_2\left(w|b_2\right)=1-F_2\left(w|b_2\right), \quad (12)$$

where $f_2\left(.|b_2\right)$ denotes the derivative of $F_2\left(.|b_2\right)$. Integrating (12) in b_2 according to $F_2\alpha_2$ over $\left[\beta_2\left(\varphi^-\left(v\right)\right),\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right]$, we find (13) below:

$$\left(w-\alpha_1\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)\right)f_2^*\left(w\right)=F_2\left(\alpha_2\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)\right)-F_2^*\left(w\right). \quad (13)$$

However, from the obvious equality $\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)=w$, we have:

$$\left(w-\alpha_1\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)\right)f_2\left(w\right)=F_2\left(\alpha_2\left(\rho^s\left(\alpha_1\left(.\right),\alpha_2\left(.\right)\right)^{-1}\left(w\right)\right)\right)-F_2\left(w\right). \quad (14)$$

Subtracting (14) from (13), we find:

$$\frac{d}{dw}\left(F_2^*(w) - F_2(w)\right) = \frac{F_2(w) - F_2^*(w)}{w - \alpha_1\left(\rho^s\left(\alpha_1(.), \alpha_2(.)\right)^{-1}(w)\right)}.$$

From this differential equation, the difference $F_2^* - F_2$ either vanishes or is of constant sign over the interval $(\varphi^-(v), \varphi^+(v))$. From this same differential equation, a constant sign implies strict monotonicity, which contradicts the equality, from Step 2, between F_2^* and F_2 at both extremities of this interval. Consequently, F_2^* and F_2 coincide everywhere and $G_2(.|.)$ is a bidding strategy of bidder 2.

FIGURE 1

Step 4 can be achieved by defining the following continuous resale-offer functions:

$$\begin{aligned} \delta_i^s(v_i; b_j) &= \max\left(\rho_i^s(v_i, \alpha_j \beta_i(v_i)), v_i\right), \text{ if } b_j \leq \beta_i(v_i); \\ &= \min\left(\rho^s(\alpha_1(b_j), \alpha_2(b_j)), \alpha_j(b_j)\right), \text{ if } b_j > \beta_i(v_i). \end{aligned}$$

$$\begin{aligned} \delta_i^b(v_i; b_j) &= \min\left(\rho_i^b(v_i, \alpha_j \beta_i(v_i)), v_i\right), \text{ if } b_j \geq \beta_i(v_i); \\ &= \max\left(\rho^b(\alpha_1(b_j), \alpha_2(b_j)), \alpha_j(b_j)\right), \text{ if } b_j < \beta_i(v_i). \end{aligned}$$

The net value functions are then as in Lemma 3 below. Since we only consider β_i, β_j from $\mathcal{E}$, we drop them from the argument of u_i^x.

Lemma 3: *Let x, y be s or b such that $x \neq y$. Let $\omega_1^y$and ω_2^y be strictly increasing. Assume bidder i expects bidder j to follow the regular bidding function β_j, for all $i \neq j$. Assume further bidder i offers the resale price according to the resale-offer function δ_i^x defined above, for all $i = 1, 2$. Then, bidder i's net-value function u_i^x, that is, the difference between his* *utility* $u_i^{x,w}$ *when winning and his utility* $u_i^{x,l}$ *when losing is as follows:*

$$\begin{aligned} & u_i^s(v_i, v_j; b_i, b_j) \\ = \; & \delta_i^s(v_i; b_j), \text{ if not larger than } v_j; \\ = \; & \delta_j^s(v_j; b_i), \text{ if not larger than } v_i; \\ = \; & v_i, \text{ otherwise;} \end{aligned}$$

$$\begin{aligned} & u_i^b(v_i, v_j; b_i, b_j) \\ = \; & \delta_i^b(v_i; b_j), \text{ if not smaller than } v_j; \\ = \; & \delta_j^b(v_j; b_i), \text{ if not smaller than } v_i; \\ = \; & v_i, \text{ otherwise;} \end{aligned}$$

for all couple of bids (b_1, b_2) *in* $[c, +\infty)^2$*, couple of use values* (v_1, v_2) *in* $[c, d]^2$*, and* $i, j = 1, 2$ *with* $i \neq j$*.*

Lemma 2 (Section 2) then extends as follows to open auctions. Notice the change of lower extremity in (i.2) with respect to the similar property Lemma 2. Here, when $x = s$ (b), the utility in case of winning (losing) $u_i^{s,w}$ $(u_i^{b,l})$ does not depend on the own bid and we may compare the expected utility to the expected utility from winning (losing) with probability one, that is, for all use values of the opponent.

Lemma 4: *Let* x, y *be* s *or* b *such that* $x \neq y$*. Let* ω_1^y *and* ω_2^y *be strictly increasing. Let* β_1, β_2 *be regular bidding functions. For all* $i = 1, 2$*, let* u_i^x *be bidder* i*'s net-value function as defined in Lemma 3. Let* i, j *be such that* $i, j = 1, 2$ *and* $i \neq j$*. Then:*

(i.1) For all (v_i, b_i) *in* $[c, d] \times [c, +\infty)$ *and all* $b_j \geq (\leq)$ b_i *if* $x = s$

(b):

$$\int u_i^x (v_i, v_j; b_i, b_j)\, dF_j (v_j|b_j) = \int u_i^x (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j)$$

(i.2) For all (v_i, b_i) in $[c, d] \times [c, +\infty)$:

$$\int_{e'}^{b_i} \int u_i^x (v_i, v_j; b_i, b_j)\, dF_j (v_j|b_j)\, dF_j \alpha_j (b_j) = \int_{e'}^{b_i} \int u_i^x (v_i, v_j; b_j, b_j)\, dF_j (v_j|b_j)\, dF_j \alpha_j (b_j),$$

where $e' = d$ if $x = s$ and $e' = c$ if $x = b$.

(ii) For all $b \geq c$:

$$\int u_i^x (\alpha_i (b), v_j; b, b)\, dF_j (v_j|b) = \rho^x (\alpha_1 (b), \alpha_2 (b)).$$

(iii) For all $b \geq c$, $\int u_i^x (v_i, v_j; b, b)\, dF_j (v_j|b)$ is nondecreasing with respect to v_i in $[c, d]$.

From Lemma 3 and Step 4, the resale price and, thus, the net value of the auction loser (winner) do not depend on his bid, when his opponent follows the equilibrium strategy and $x = s\ (b)$. (i.1) and (i.2) follow.

(ii) holds true because resale occurs with probability one when both bidders submit the same bid b (and $\alpha_1 (b) \neq \alpha_2 (b)$). In fact, assume, for example, $x = s$ and $\alpha_2 (b) > \alpha_1 (b)$. Then, according to the revised beliefs, bidder 1's use value is $\alpha_1 (b)$ and the minimum of the support of bidder 2's use value is $\rho^s (\alpha_1 (b), \alpha_2 (b))$. With probability one bidder 1's resale offer $\rho^s (\alpha_1 (b), \alpha_2 (b))$ is accepted by bidder 2 and is equal to the bidders' net values.

Bidder i obtains the conditional net expected payoff $\int u_i^x(v_i, v_j; b_j, b_j)\, dF_j(v_j|b_j)$ if he observes b_j and proposes his optimal resale price when he is the price setter. It is thus the maximum of the net expected payoff he obtains when he proposes p_i, over all possible resale prices p_i. Since, for any fixed p_i, his net expected payoff is nondecreasing in his use value v_i, so will his optimal net expected payoff and (iii) follows.

From Lemma 4, proceeding as in Section 2, we obtain the same sets of optimal bids. Since those sets are the supports of the bidding strategies $G_i(.|.)$, $i = 1, 2$, $\mathcal{E}'$ in Corollary 3 is a regular equilibrium of the open auction.

From Lemma 4 (i.1), given the opponent's bid, a bidder's bid does not affect his net value for winning. In the second-price auction, a bidder's bid has obviously no effect on the auction price when he wins. From Lemma 4 and the equality $\rho^x(\alpha_1(b), \alpha_2(b)) = b$, any of a bidder's equilibrium bids in the second-price auction wins against bids that contribute nonnegatively to his net expected payoff and loses against those that contribute nonpositively. He has thus no incentive to change his own bid, even if he learns his opponent's bid, and we have proved Corollary 3 (iii).

The final allocation is the same in $\mathcal{E}'$ as in $\mathcal{E}$. Assume, for example, $x = s$ and bidder 1's use value v_1 is such that $\varphi(v_1) \geq v_1$, where $\varphi = \alpha_2\beta_1$. Then, bidder 1 bids $\beta_1(v_1)$. If λ_φ^s is as defined in Corollary 2 (Section 2), we have $v_1 \leq \lambda_\varphi^s(v_1) = \rho^s(v_1, \varphi(v_1)) \leq \varphi(v_1)$. If $v_2 \leq \lambda_\varphi^s(v_1)$, Step 2 implies that bidder 2 with use value v_2 bids at most $\max\left(\beta_1(\varphi^-(v_1)), \rho^s(\alpha_1(.), \alpha_2(.))^{-1}(v_2)\right)$, which is not larger than $\beta_1(v_1)$. Consequently, bidder 2 loses the auction and refuses bidder 1's offer. If $v_2 \geq \lambda_\varphi^s(v_1)$, bidder 2 accepts bidder 1's resale offer when bidder 1 wins and there is no profitable resale when bidder 2 wins.

From Myerson (1981) (Lemma 2, p.63), the interim expected payoffs are the same in $\mathcal{E}'$ as in $\mathcal{E}$. Because, by construction, the marginal bid distributions are the same, we have proved Corollary 3 (i).

Since, from Step 1, a bidder who can be a price setter at resale submits the same bid, his interim probability of winning is also the same in both equilibria. From Step 2, when resale is possible, the price setter makes the same offer. In order to generate the same interim expected payoffs, the probabilities of resale must be the same in both equilibria and we have proved Corollary 3 (ii).

4. Comparative Statics

From Myerson (1981), in equilibrium, a couple of use values (v_1, v_2) contributes to the auctioneer's expected revenues the (buyer's) virtual use value of the eventual owner. Without loss of generality, assume that $\omega_1^b(c) \geq \omega_2^b(c)$. The function $\psi = \left(\omega_2^b\right)^{-1} \omega_1^b$ then determines the final allocation

that maximizes the expected revenues. Its definition is equivalent to (15) below:

$$\omega_2^b(\psi(v)) = \omega_1^b(v), \quad (15)$$

for all v in $[c, d]$. From this definition, $\psi(d) = d$.

In Lemma 5 below, λ_φ^s (λ_φ^b), where $\varphi = F_2^{-1}F_1$, is the final equilibrium allocation, defined in Corollary 2 (Section 2), of the first-price auction where the auction winner (loser) is the price setter at resale.

Lemma 5: *Let $\omega_1^s, \omega_2^s, \omega_1^b, \omega_2^b$ be strictly increasing. Let φ be equal to $F_2^{-1}F_1$. Assume $\omega_1^b(c) \geq \omega_2^b(c)$. Then, for all v in $[c, d]$:*

$$\lambda_\varphi^s(v) > (<) \lambda_\varphi^b(v) \text{ if and only if } \lambda_\varphi^s(v) < (>) \psi(v).$$

The proof is straightforward. Assume, for example, $\varphi(v) > v$. From Corollary 2 (Section 2) and $\varphi = F_2^{-1}F_1$, we have $\lambda_\varphi^s(v) = \rho^s\left(v, F_2^{-1}F_1(v)\right)$

and $v = \rho^b \left(F_1^{-1} F_2 \left(\lambda_\varphi^b (v)\right), \lambda_\varphi^b (v)\right)$, that is:

$$v = \lambda_\varphi^s (v) - \frac{F_1 (v) - F_2 \left(\lambda_\varphi^s (v)\right)}{f_2 \left(\lambda_\varphi^s (v)\right)}, \quad (16)$$

$$v = \lambda_\varphi^b (v) + \frac{F_2 \left(\lambda_\varphi^b (v)\right) - F_1 (v)}{f_1 (v)}. \quad (17)$$

From the definitions of ω_1^b, ω_2^b, (16) is equivalent to (18) below:

$$\omega_2^b \left(\lambda_\varphi^s (v)\right) = \omega_1^b (v) + (1 - F_1 (v)) \left(\frac{1}{f_1 (v)} - \frac{1}{f_2 \left(\lambda_\varphi^s (v)\right)}\right). \quad (18)$$

If $\lambda_\varphi^s (v) < \psi (v)$, (15) and (18) imply $f_1 (v) > f_2 \left(\lambda_\varphi^s (v)\right)$. Since $F_1 (v) - F_2 \left(\lambda_\varphi^s (v)\right) > 0$, (16) then implies:

$$v < \lambda_\varphi^s (v) - \frac{F_1 (v) - F_2 \left(\lambda_\varphi^s (v)\right)}{f_1 (v)}. (19)$$

Combining (19) with (17) and rearranging give:

$$\lambda_\varphi^b (v) + \frac{F_2 \left(\lambda_\varphi^b (v)\right)}{f_1 (v)} < \lambda_\varphi^s (v) + \frac{F_2 \left(\lambda_\varphi^s (v)\right)}{f_1 (v)},$$

which immediately implies $\lambda_\varphi^b (v) < \lambda_\varphi^s (v)$. Lemma 5 is proved once it is noticed that all our implications are actually equivalences.

From Lemma 5, wherever the two allocations λ_φ^b and λ_φ^s differ, λ_φ^s makes the better choice by choosing the bidder with the higher virtual use value. From Myerson (1981), higher expected revenues accrue to the auctioneer under λ_φ^s and we have Corollary 4 below. Contrary to the case of resale under complete information of Gupta and Lebrun (1999), there is in general no relation of stochastic dominance between the bid distributions under the two bargaining procedures[8].

Corollary 4: *Let $\omega_1^s, \omega_2^s, \omega_1^b, \omega_2^b$ be strictly increasing. Let R^s (R^b) be the auctioneer's expected revenues at the unique regular equilibrium of the first-price auction when the auction winner (loser) is the price-setter at resale. Then, we have:*

$$R^s \geq R^b.$$

The regular equilibrium of the second-price auction that is constructed as in Theorem 1 from $\varphi = F_2^{-1}F_1$ allocates the item as the unique equilibrium of the first-price auction does and hence gives the same expected revenues. In the symmetric case $F_1 = F_2$, the functions λ_φ^s, λ_φ^b and ψ are all equal to the identity function and the equilibrium of the first-price auction is an optimal mechanism. This is no longer the case in the asymmetric case $F_1 \neq F_2$. This point is most clearly made under the assumption of differentiability of ψ. Assume that there exists v such that $\varphi(v) > v$. Suppose λ_φ^s and ψ or, equivalently, from Lemma 5, λ_φ^b and ψ are identical over $(\varphi^-(v), \varphi^+(v))$. From (18) and (16), $f_1(w) = f_2(\psi(w))$ and $F_1(w) - F_2(\psi(w)) = f_1(w)(\psi(w) - w)$, for all w in $(\varphi^-(v), \varphi^+(v))$. Because its derivative then vanishes, $(F_1(w) - F_2(\psi(w)))(\psi(w) - w)$ is constant over this interval. However, this is impossible since it tends towards zero at the extremities.

In the asymmetric case, there thus exists an interval where λ_φ^x is everywhere different from ψ, for all $x = s, b$. By slightly moving φ over this interval, while keeping it continuous and strictly increasing, towards and away from ψ, λ_φ^x will move in the same direction and we have Corollary 5 below.

[8] From Section 2, the bids are identically distributed according to G^x such that $(G^x)^{-1}(q) = \rho^x\left(F_1^{-1}(q), F_2^{-1}(q)\right)$, for all q in $[0,1]$. Consider the case where there exists q, i, j such that $F_i^{-1}(q) < F_j^{-1}(q)$, F_i is strictly convex over $\left[F_i^{-1}(q), F_j^{-1}(q)\right]$, and F_j is strictly concave over the same interval. Then, the definitions of ρ^s and ρ^b easily imply $\rho^b\left(F_1^{-1}(q), F_2^{-1}(q)\right) > \frac{F_1^{-1}(q)+F_2^{-1}(q)}{2} > \rho^s\left(F_1^{-1}(q), F_2^{-1}(q)\right)$, and G^s does not stochastically dominate G^b.

Corollary 5: *Let $\omega_1^s, \omega_2^s, \omega_1^b, \omega_2^b$ be strictly increasing and let ψ be differentiable. Then:*

(i) When $F_1 = F_2$, the unique regular equilibrium of the first-price auction gives revenues equal to the maximum revenues at regular equilibria of the second-price auction.

(ii) When $F_1 \neq F_2$, the revenues from the unique regular equilibrium of the first-price auction are strictly smaller than the revenues at some equilibria of the second-price auction and strictly larger than the revenues at some others.

5. Conclusion

We established a link between the common value model and the independent private value model with resale. From this link, we characterized an infinity of equilibria of the second-price auction with resale and a unique equilibrium of the first-price auction with resale. For every equilibrium of any auction without disclosure of the bids, we constructed an equivalent equilibrium of the auction with disclosure of the bids. All our equilibria of the second-price auction with bid disclosure satisfy a no-regret property: after learning the bids, no bidder regrets his own.

We showed an equilibrium of the second-price auction that gives the same expected revenues as the equilibrium of the first-price auction. With heterogeneous bidders, we showed equilibria of the second-price auction that give strictly higher expected revenues.

For the first-price auction, we proved that the expected revenues are higher when the price setter at resale is the auction winner.

Appendix 1

Lemma A1: *Let x, y be s or b and such that $x \neq y$. Let ω_1^y and ω_2^y be strictly increasing. Let β_1, β_2 be regular bidding functions and let u_1^x, u_2^x be defined as in Lemma 1. Let $i, j = 1, 2$ be such that $i \neq j$. Then, for all $b_i \geq c$ and v_i in $[c, d]$, the function $\widetilde{u}_i^x (v_j) = u_i^x \left(v_i, v_j; b_i, \beta_j (v_j); \beta_i, \beta_j\right)$ of v_j is continuous with respect to v_j at $v_j = \alpha_j (b_i)$ and almost all other v_j in $[c, d]$.*

Proof: From the definition of u_i^x in Lemma 1 and the continuity of ρ^x and α_i, $\widetilde{u}_i^x$ is continuous at v_j if $v_j \neq \rho_i^x (v_i, \max (v_i, \alpha_j (b_i)))$. Assume v_j in $[c, d]$ is such that $v_j = \alpha_j (b_i)$ and $v_j = \rho_i^x (v_i, \max (v_i, \alpha_j (b_i)))$. Then, $v_j = v_i$. Since the function $\widetilde{u}_i^x$ always lies between v_i and v_j, it is continuous if $v_j = v_i$ and Lemma A1 follows. ||

Proof of Lemma 2:

Proof of (i): Through the change of variables $w_j = \alpha_j (b_i)$, (5) implies

$$w_j \in \arg \max_{w_j' \in [c,d]} \int_e^{w_j} u_i^x \left(v_i, v_j; \beta_j \left(w_j'\right), \beta_j (v_j); \beta_i, \beta_j\right) dF_j (v_j), \text{(A1.1)}$$

for all w_j in $[c, d]$. For all w_j' in $[c, d]$, the objective function in (A1.1), as an integral, is absolutely continuous with respect to w_j and, from Lemma A1 and the continuity of f_j, the integrand is continuous with respect to v_j almost everywhere in $[c, d]$. Consequently, the derivative of the objective function at w_j exists and is equal to $u_i^x \left(v_i, v_j; \beta_j \left(w_j'\right), \beta_j (v_j); \beta_i, \beta_j\right) f_j (w_j)$, for almost all w_j in $[c, d]$. Since u_i^x and f_j are bounded, the assumptions of a variant[9] of Theorem 2 in Milgrom and Segal (2002) are satisfied. From this variant and the change of variables $w_j = \alpha_j (b_i)$, (i) follows for all b_i in $\left[c, \beta_j (d)\right]$.

(i) reduces to the trivial equality $0 = 0$ when $x = b$ and $b_i \geq \beta_j (d)$.

[9] This is the variant (which can be proved as Theorem 2 in Milgrom and Segal (2002) from their Theorem 1) where the requirement that $f (x, .)$ be differentiable for all $x \in X$ is replaced by the requirement that $f (x^* (t), .)$ be differentiable, for any selection $x^* (.) \in X^* (.)$ and almost all $t \in (0, 1)$.

When $x = s$, (5) implies that the objective function in (5) is constant with respect to $b_i' \geq \beta_j(d)$, for all $b_i \geq \beta_j(d)$. (i) then follows.

Proof of (ii): If $\alpha_1(b) = \alpha_2(b)$, Lemma 1 implies $u_i^x(\alpha_1(b), \alpha_2(b); b, b; \beta_1, \beta_2) = \alpha_i(b)$. Since $\rho^x(\alpha_1(b), \alpha_2(b)) = \alpha_i(b)$, (ii) follows. If $\alpha_1(b) \neq \alpha_2(b)$, $\min(\alpha_1(b), \alpha_2(b)) < \rho^x(\alpha_1(b), \alpha_2(b)) < \max(\alpha_1(b), \alpha_2(b))$ and (ii) follows from Lemma 1.

Proof of (iii): Wen $x = s$ (b), $u_i^x\left(v_i, \alpha_j(b); b, b; \beta_i, \beta_j\right)$ is, from Lemma 1, equal to $\rho_i^x(v_i, \alpha_j(b))$ if $v_i < (>) \alpha_j(b)$, to $\alpha_j(b)$ if $v_i = \alpha_j(b)$, and to $\min\left(v_i, \rho_j^x(\alpha_j(b), \max(\alpha_j(b), \alpha_i(b)))\right)$ $\left(\max\left(v_i, \rho_j^x(\alpha_j(b), \min(\alpha_j(b), \alpha_i(b)))\right)\right)$ if $v_i > (<) \alpha_j(b)$. (iii) follows. ||

Appendix 2

Lemma A2: *Let i,$j = 1, 2$ be such that $i \neq j$. If ω_j^s is strictly increasing, then the left-hand partial derivative $\frac{\partial_l}{\partial v_i}\rho^s(v, v)$ exists and*

$$\frac{\partial_l}{\partial v_i}\rho^s(v, v) = \frac{1}{2},$$

for all v in $(c, d]$. If ω_j^b is strictly increasing, then the right-hand partial derivative $\frac{\partial_r}{\partial v_i}\rho^b(v, v)$ exists and

$$\frac{\partial_r}{\partial v_i}\rho^b(v, v) = \frac{1}{2}$$

for all v in $[c, d)$.

Proof: We prove the statement about ρ^s. The statement about ρ^b can be similarly proved. Let v_1, v_2 be such that $c < v_i < v_j \leq d$, with $i \neq j$. Subtracting the definition (1) of $\rho(v_1, v_2)$ from v_j and dividing by $v_j - v_i$, we find:

$$1 = \frac{v_j - \rho(v_1, v_2)}{v_j - v_i}\left(1 + \frac{1}{f_j(\rho(v_1, v_2))}\frac{F_j(v_j) - F_j(\rho(v_1, v_2))}{v_j - \rho(v_1, v_2)}\right).$$

From the continuity of f_j at v_j and the continuity of ρ, $f_j(\rho(v_1, v_2))$ tends towards $f_j(v_j)$, when v_i tends towards v_j from below. Since the derivative of F_j at v_j exists and is equal to $f_j(v_j)$, the limit of the ratio $\frac{F_j(v_j)-F_j(\rho(v_1,v_2))}{v_j-\rho(v_1,v_2)}$ is equal to $f_j(v_j)$. Consequently, the factor between parentheses in the equation above tends towards 2 and the lemma follows. ||

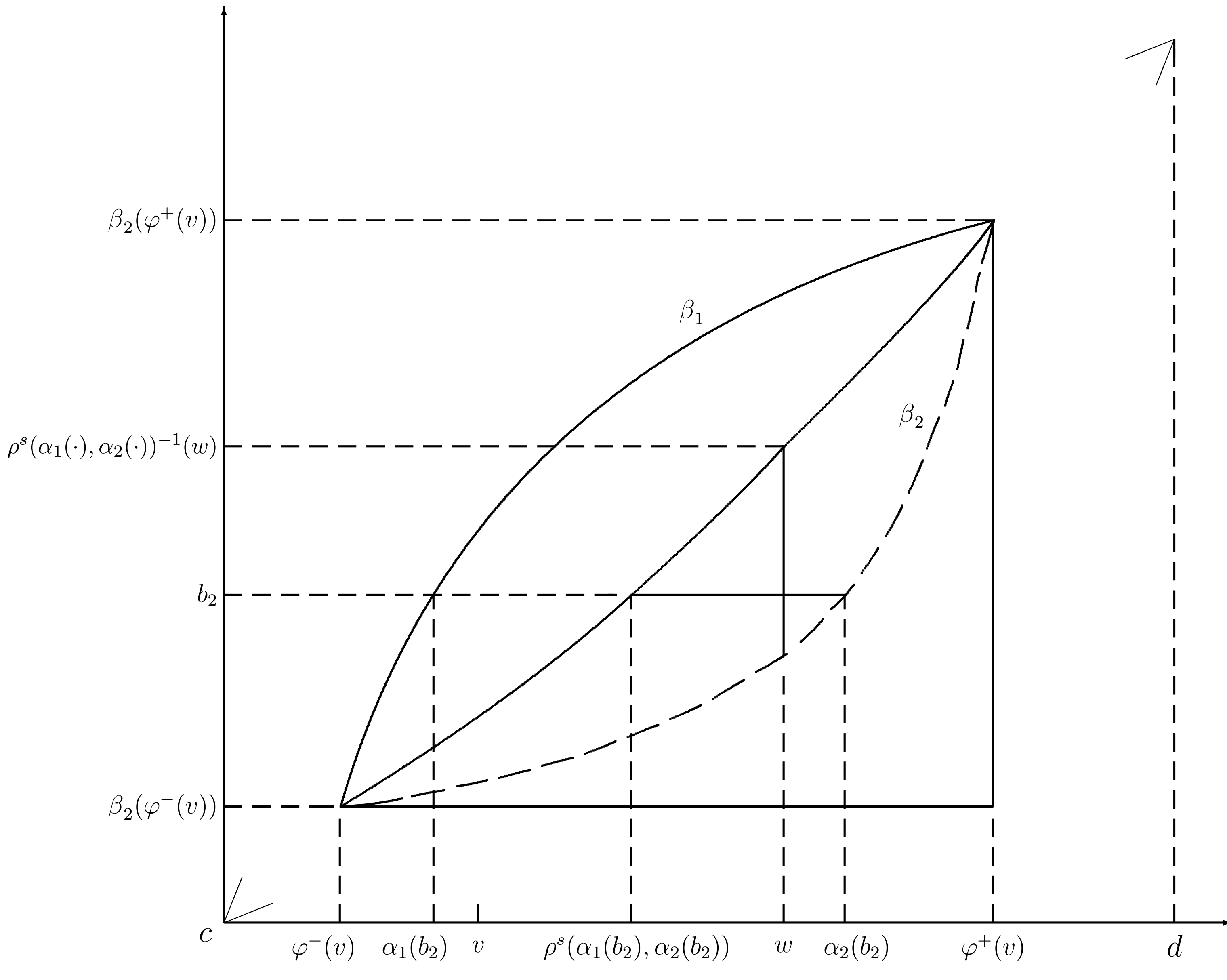
Proof that (11) is a cumulative distribution function:

As indicated in the main text, we only need to prove that (11) tends towards one as v_2 tends towards $\varphi^+(v)$. Since it is well known that the integral $\int_{\rho^s(\alpha_1(b_2),\alpha_2(b_2))}^{\varphi^+(v)} \frac{1}{\varphi^+(v)-w} dw$ diverges and since $\frac{w-\alpha_1\left(\rho^s(\alpha_1(.),\alpha_2(.))^{-1}(w)\right)}{\varphi^+(v)-w} = -1 + \frac{\varphi^+(v)-\alpha_1\left(\rho^s(\alpha_1(.),\alpha_2(.))^{-1}(w)\right)}{\varphi^+(v)-w}$, we will be done once we prove that $\frac{\varphi^+(v)-w}{\varphi^+(v)-\alpha_1\left(\rho^s(\alpha_1(.),\alpha_2(.))^{-1}(w)\right)}$ is bounded away from zero, as w tends towards $\varphi^+(v)$ from below, or, equivalently, through the change of variables $v_1 = \alpha_1\left(\rho^s(\alpha_1(.), \alpha_2(.))^{-1}(w)\right)$, that $\frac{\varphi^+(v)-\rho^s(v_1,\varphi(v_1))}{\varphi^+(v)-v_1}$ is bounded away from zero, as v_1 tends towards $\varphi^+(v)$ from below.

However, this last ratio is equal to the sum $\frac{\varphi^+(v)-\rho^s\left(v_1,\varphi^+(v)\right)}{\varphi^+(v)-v_1} + \frac{\rho^s\left(v_1,\varphi^+(v)\right)-\rho^s(v_1,\varphi(v_1))}{\varphi^+(v)-v_1}$. The second term is nonnegative and, from $\frac{\partial_l}{\partial v_2}\rho^s(\varphi^+(v), \varphi^+(v)) = \frac{1}{2}$ (Lemma A2), the first term tends towards $\frac{1}{2}$. Consequently, it is bounded away from zero and the proof is complete.

The proof for the other bargaining procedure is similar and makes use of $\frac{\partial_r}{\partial v_2}\rho^b(\varphi^-(v), \varphi^-(v)) = \frac{1}{2}$. ||

Figure 1



$\beta_2(\varphi^+(v))$
β_1
β_2
$\rho^s(\alpha_1(\cdot),\alpha_2(\cdot))^{-1}(w)$
b_2
$\beta_2(\varphi^-(v))$
c
$\varphi^-(v)$
$\alpha_1(b_2)$
v
$\rho^s(\alpha_1(b_2),\alpha_2(b_2))$
w
$\alpha_2(b_2)$
$\varphi^+(v)$
d

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