The Dynamics of Hate and Violence

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November 15, 2013

Abstract

This paper provides a simple dynamic model that explores the interdependence and dynamic properties of hate, violence and economic well-being. It shows that a time-dependent economic growth process that affects the evolution of hate can yield a long run steady state, but this steady state will not be free of hate and violence. Moreover, we show that better (long run) economic conditions do not necessarily result in lower equilibrium levels of hate and violence. We show that, under reasonable conditions, cycles of hate and violence cannot occur and consequently, the dynamics of hate and violence cannot give rise to cyclical patterns of economic well-being. While both stable and unstable equilibria are possible, the most likely equilibrium is unstable (a saddle point). Using a linear example, we show that instability becomes more likely with an increase in the: responsiveness to economic condition and violence; willingness to forgive; marginal cost of violence and "length" of a country's memory.

KEYWORDS: Hate, Violence, Dynamics, Steady State, Stability, Genuine Peace.

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†I wish to thank Barry Smith and Jianhong Wu for useful comments and suggestions.

1 Introduction

Hate and violence have been a common feature of human history. Secular and religious scholars, as well as leaders and warriors, have discussed their nature throughout history. As far back as the 5th century BC, Thucydides wrote about the nature of violence, observing that wars lead to even worse wars. The Bible, likewise, recognized that[1] "violence begets violence." The relationship between hate and violence is complex. What is clear, however, is that the two are very closely intertwined: they affect and are affected by each other. Making matters more complex is the fact that they are both affected by other factors which, in turn, impact their evolution. In the aftermath of 9/11, hate and violence have become the subject of renewed endless debates, as well as extensive academic research. Indeed, there is now a vast literature on the subject encompassing historical, philosophical, social, religious, psychological, political, economic and cultural aspects of hate and violence.[2]

Naturally, in the academic literature, each discipline has its own focus, tools and perspectives; possibly also its own pre-conceptions. Thus, for example, many sociologists, psychologists and political scientists explain hate by underlying "root causes".[3] Economists, on the other hand, tend to explain all phenomena, hence also hate and violence, as an outcome of underlying optimal decision-making processes (in addition to root causes).[4] Behaviour by individuals or governments is, therefore, explained as the outcome of an underlying strategic "game".[5] Interestingly, though, even non-economists often argue that phenomena like hate, extremism and violence can be explained by economic considerations such as competition over scarce resources.[6] Here too, government policies, including incitement in the face of such rivalry, are viewed as part of an overall strategy.

The purpose of this paper is to develop a simple model that can explain the dynamic properties of hate and violence. Instead of focusing on possible root causes, or strategic policy determinants of hate, we focus on its evolution and properties. We do not argue that root causes or strategic considerations do not play a role: we acknowledge that they do. But, although they may affect hate, the level of hate and its evolution are not a matter of choice by individuals; they are governed by motion (evolution) equations. Even strategic policy makers must take these motion equations into account. We can think of strategic policy makers as using these motion equations to "their advantage". That is, a strategic policy is a way of "manufacturing" root causes: it can induce hate directly (e.g., by incitement), or indirectly (by affecting the root causes). To paraphrase

[1] Matthew 26:52, King James Bible version: "for all they that take the sword shall perish with the sword".

[2] For example, Sternberg and Sternberg (2008) provide a comprehensive discussion of the psychological aspects of hate, Nozick (1997) and Breton (2002) provide a general discussion of political, economic and philosophical aspects of extremism.

[3] For examples of studies of root causes see Blomberg and Hess (2002), Blomberg et al. (2004), Bandarage (2004) and Sandler and Enders (2004).

[4] See discussions in Glaeser (2005), Cameron (2009).

[5] See for example, Schelling (1960), Atkinson, Sandler and Tschirhart (1987), Enders and Sandler (2006), Sørli, Gleditsch and Strand (2005), Collier and Hoeffler (1998), Esteban and Ray (1999), Wintrobe (2006), Appelbaum and Katz (2007), Appelbaum (2008), Medoff (1999).

[6] See for example, Sternberg and Sternberg (2008), Piazza (2006), Eizenstat Porter and Weinstein, (2005) and Pape (2003). See also Sherif (1966), who proposes a "realistic conflict theory".

Herman and Chomsky (1988), these strategic policies serve to "manufacture dissent". Nevertheless, since the evolution of hate is still governed by the motion rules, strategic policies must also take these rules into account.[7]

The paper examines the dynamics of hate and violence in a conflict between two countries (rival groups, communities, etc.) that have a history of conflict, hate and violence. The history of the relationship may be explained by ethnic, racial or religious conflicts. It may also be due to geographical, ideological, or economic conflicts. In general, the root causes of the conflict reflect a combination of these underlying factors. But, regardless of the underlying root causes, within such a relationship, hate and violence are clearly interdependent: they affect and are affected by each other. In fact, both the literature on the psychology of hate and the literature on the political/sociological theory of conflicts, recognize that violence and hate may result in a "vicious circle": violence breeds hate, but hate in turn breeds violence.[8] Moreover, economic considerations also become part of this vicious circle. Economic well-being affects the evolution of hate directly (presumably, an improvement in economic well-being mitigates the evolution of hate). But, economic well-being itself is affected by costly violence,[9] which in turn is affected by hate.

These interdependent relationships and their effect on the evolution of hate are captured in this paper within a system of differential equations. We use the model to study the dynamic properties of hate and violence. We begin by defining an ideal state of "genuine peace",[10] in which there is neither hate, nor violence. We then show that, given a time-dependent economic growth process (that affects the evolution of hate), a long run steady state is possible but, in general, such steady state will not be characterized by genuine peace. Moreover, we show that a better long run economic environment does not necessarily result in lower equilibrium levels of hate and violence. Next, we examine the stability properties of the (hate and violence) equilibrium. First, we show that, under reasonable conditions (when both rivals are "congruent", or reciprocating countries, who are attuned to each other's nature), cycles of hate and violence cannot occur. Consequently, the dynamics of hate and violence in itself cannot give rise to cyclical patterns of (net) economic well-being; such cyclical patterns can only arise due to cyclical patterns of gross growth rates. We demonstrate that for cycles to occur, we would need to have a case where, for one and only one country, reciprocity (congruency) does not hold. Second, we show that, while it is possible to have either stable, or unstable equilibria, under reasonable conditions it is more likely to have unstable ones. Specifically, the most likely outcome is a saddle point. Finally, we provide an example of a linear system that can be used to study the effects of changes in the parameters on the stability of the system. We show that instability becomes more likely if there is an increase in the responsiveness to economic condition and violence; the unwillingness to forgive; the marginal cost of

[7] A differential game model, with strategic governments, is currently pursues by the author.

[8] See Minow (2002) Levin and Rabrenovic (2001).

[9] It is also possible that "net economic well-being" is affected by captured resource hence giving rise a strategic motive for the conflict.

[10] Discussed, for example, in Levi (2008).

violence and the "length" of a country's memory.

2 The Model

Consider the dynamic relationship between two countries (rival groups, communities, etc.) that have a history of conflict, hate and violence. To model the interdependence of the factors that affect the evolution of hate, we begin by looking at the determinants of violence. Let Country $i's$ hate toward Country j, $(i \neq j)$, at each point in time, t, be given by $h^i(t)$. We allow for the possibility the hate may be negative. In such a case we can think of it as "love". We view hate as a state (stock) variable that summarizes the state of a country's antagonism in the conflict. Let Country $i's$ violence toward Country j, $(i \neq j)$, at each point in time, t, be given by $v_i(t)$. We allow for the possibility the violence may be negative, in which case we can think of it as "benevolence". We view violence as a "flow variable". We assume that a country's violence toward its rival depends on three variables: its hate toward the rival, the rival's level of hate and the rival's level of violence. This is described by the following two continuously differentiable functions:

$$\begin{aligned} v_1(t) &= V^1[h_1(t), h_2(t), v_2(t)] \\ v_2(t) &= V^2[h_1(t), h_2(t), v_1(t)] \end{aligned} \tag{1}$$

We assume that when hate in both countries is zero, the solution to the two equations is $v_1 = v_2 = 0$, namely: $V^i[0,0,0] = 0$, $i = 1, 2$.

Country $i's$ violence toward Country j $(\neq i)$ is assumed to be an increasing function of its hate, the other country's hate and the other country's violence:

$$\begin{aligned} \frac{\partial V^i[h_1(t), h_2(t), v_2(t)]}{\partial h_j} &> 0,\ i, j = 1, 2 \\ \frac{\partial V^i[h_1(t), h_2(t), v_2(t)]}{\partial v_j} &> 0,\ i, j = 1, 2,\ i \neq j, \end{aligned} \tag{2}$$

Let us now examine the role of economic considerations. Consider some measure of economic performance, at time t, in Country i. For example, this may be a measure of Country $i's$ GDP. We assume that any type of violence in the conflict (regardless of its source) has an economic cost. Denote *gross* and *net* GDP in Country i, at time t as $x_i(t)$ and $y_i(t)$, respectively. Suppose that a fraction of gross GDP is lost due to violence in the conflict. Let this fraction, denoted as c^i, be captured by the continuously differentiable cost function,

$$c^i = C^i[v_1(t), v_2(t)]),\ i = 1, 2$$

We assume that the cost function is increasing in violence (in either country), but when there is no violence the costs are zero. Namely,

$$\frac{\partial C^i[v_1(t), v_2(t)]}{\partial v_j} > 0,\ i, j = 1, 2 \tag{3}$$

$$C^i[0,0] = 0, \ i = 1,2$$

Thus, *net* GDP is given by:[11]

$$y_i[v_1(t), v_2(t), t] \equiv x_i(t)(1 - C^i[v_1(t), v_2(t)]), \ i = 1,2 \tag{4}$$

Now, define the *net* economic growth rate in Country i as:

$$w_i[v_1(t), v_2(t), t] \equiv d\ln[y_i[v_1(t), v_2(t), t]]/dt = \tag{5}$$

$$d\ln[x_i(t)]/dt + d\ln[1 - C^i[v_1(t), v_2(t)]]/dt \equiv g_i(t) + r^i[v_1(t), v_2(t), t]$$

where $g_i(t)$ is the gross rate of economic growth and $r^i[v_1(t), v_2(t), t]$ is the rate of growth in the fraction of GDP that is *not* lost due to violence $(1 - C^i)$. Note that since $\partial C^i[v_1(t), v_2(t)]/\partial v_j > 0$, we have:

$$\frac{\partial r^i[v_1(t), v_2(t), t]}{\partial v_j} < 0, \ i, j = 1,2 \tag{6}$$

Thus, since in the absence of violence, the net and gross rates of economic growth are the same, we have:

$$w_i[0,0,t] = g_i(t) \tag{7}$$

Let us assume that the gross rate of economic growth converges to some "long run" value of ,

$$g_i^* = \lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} w_i[0,0,t].$$

We can now turn to the evolution of hate. We assume that the evolution of hate depends on the countries': (i) levels of hate (ii) levels of violence and (iii) net economic growth. The evolution of hate can, therefore, be described by the following two differential equations:

$$\begin{aligned} \frac{dh_1}{dt} &= H^1(h_1, h_2, v_1, v_2, w_1) \\ \frac{dh_2}{dt} &= H^2(h_1, h_2, v_1, v_2, w_1) \end{aligned} \tag{8}$$

where H^1 and H^2 are continuously differentiable functions and where, for notational simplicity, the time variable, t, is dropped for the rest of the paper whenever it is not required. The complexity of the dynamics of hate is due to the fact that it depends on hate, violence and economic conditions, but levels violence themselves are determined simultaneously and depend on hate levels and in addition they affect net economic conditions.

Let us now consider the likely properties the evolution of hate equations. In general, the response of a country's evolution of hate to its rival's hate falls into one of two categories: it may, or may not be reciprocating

[11] Note that it may be possible to consider an alternative scenario where $v_i(t)$ reflects, at least partially, an attempt by country i to gain resources from country j. In other words, at least some of the violence may "rational", in the sense that it is motivated by "strategic aggression". This, requires a differential game framework which will be pursued in a separate paper.

$(\partial H^i/\partial h_{j\neq i} < 0$, or $\partial H^i/\partial h_{j\neq i} > 0)$. Essentially, this captures the country's degree of congruence (being attuned to the nature of its rival). Similarly, the response of a country's evolution of hate to its own hate falls into one of two categories: it may have a long, or a short "memory" $(\partial H^i/\partial h_i < 0$, or $\partial H^i/\partial h_i > 0)$. There are, therefore, four possible combination to consider. These cases are shown in Table 1:

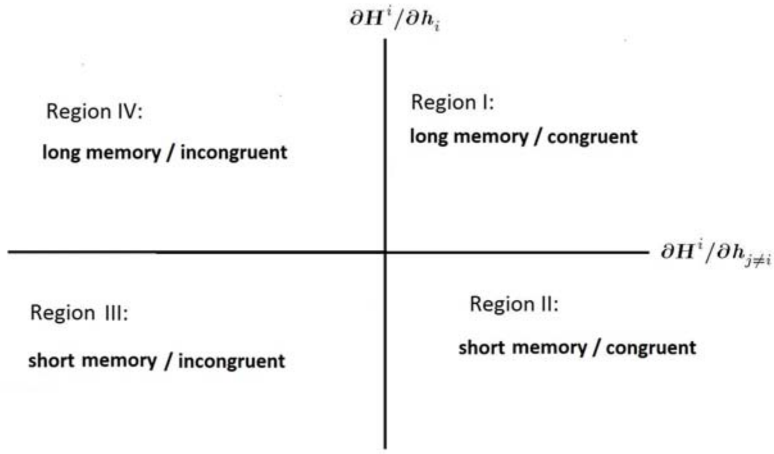

Table 1: The Nature of Country i - Effects of Hate

Table 1 above describes, what we refer to as, a country's *intrinsic type.* As we will show later, however, a country may have a "perceived" type which is different than its intrinsic one. This may occur for the following reason. Violence and economic condition depend on hate and consequently the effects of a change in hate has both direct and indirect effects. The direct effect is what we referred to above as the intrinsic type (captured by $\partial H^i/\partial h_j$). The indirect effect captures the effect of a change in hate on the evolution of hate through its effects on violence and economic conditions (or other, so called, "root causes"). The overall, or "perceived" effect is the sum of the two, so there is no reason why the overall effect should have the properties (say, sign) as the intrinsic type. This will be shown below.

As for the effects of violence on a country's evolution of hate, here too, we may have four cases. A country may be "masochistic", or vengeful $(\partial H^i/\partial v_{j\neq i} < 0$, or $\partial H^i/\partial v_{j\neq i} > 0)$ with respect to violence by its rival. Moreover, a country's own violence may give rise to a "need to justify" its actions by minimizing cognitive dissonance, hence effecting the evolution of hate positively $(\partial H^i/\partial v_i > 0)$. On the other hand, it may exhibit dissonance "affinity", or incongruence, so that its own violence and the evolution of its hate are negatively related $(\partial H^i/\partial v_i < 0)$. These cases are shown in Table 2:

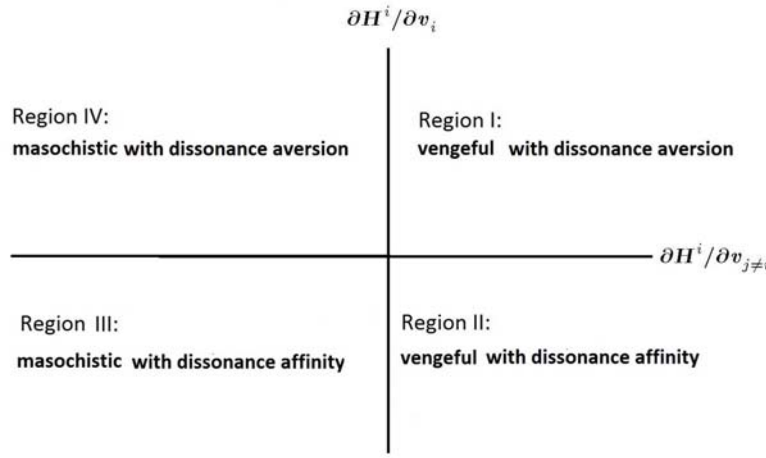

Table 2: The Nature of Country i - Effects of Violence

Although, there are several possible configurations, not all are equally likely. Specifically, it does not seems likely that an increase in the rival's levels of hate, or violence would have a negative effect on a country's evolution of hate. Thus, we can expect to have,[12]

$$\begin{aligned} \frac{\partial H^i[h_1, h_2, v_1, v_2, w_1]}{\partial h_j} &> 0, \quad i, j = 1, 2, \ i \neq j \\ \frac{\partial H^i[h_1, h_2, v_1, v_2, w_2]}{\partial v_j} &> 0, \quad i, j = 1, 2, \ i \neq j \end{aligned} \tag{9}$$

This eliminates regions III and IV in Tables 1 and 2. Furthermore, the literature on the psychology of hate has found that, as a result of the need to minimize cognitive dissonance, violence against a rival will actually lead to an increase in hate toward the rival.[13] This eliminates regions II and III in Table 2.[14] Thus, we assume that:

$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_i]}{\partial v_i} > 0, \quad i = 1, 2 \tag{10}$$

The effect of a country's own hate on the evolution of its hate, however, is not that clear. On the one hand, as is common in macroeconomic models, there may be persistence, or inertia. On the other hand, there may be "depreciation", or willingness to forget.[15] Hence, it is not clear how the current level of a country's own hate affects its evolution of hate; $\partial H^i/\partial h_i$ may be positive, negative, or zero.[16] Altogether, therefore, regions I, II in Table 1 and region I in Table 2 are likely to arise. If we are willing to allow for remorsefulness, then it is also possible to have a remorseful, but vengeful country.

Finally, it is well recognized in the literature that economic well-being affects the nature, prevalence and severity of conflicts.[17] Thus, we assume that net economic growth in a country has a negative effect on a

[12] This is, therefore, related to positive feedback natural systems. See DeAngelis, et. al. (1985).

[13] See Festinger and Carlsmith, (1959), Sternberg and Sternberg (2008).

[14] Note that if region II in Table 2 is not eliminated, we would end up with a case of a vengeful, but "remorseful" country.

[15] This is common in macroeconomic models and in models of love. See, for example, Strogatz (1994), Rinaldi (1998), and Sprott (2004).

[16] See Levy (2008) for example.

[17] For example, Muller and Weede (1990) and Blomberg, Hess and Weerapana (2004) find that high levels of economic well-being

country's evolution of hate.[18] In other words, we have:

$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_i]}{\partial w_i} < 0, \quad i = 1, 2$$

Since $w_i = g_i + r^i$, this implies that:

$$\frac{\partial H^i[h_1, h_2, v_1, v_2, w_i]}{\partial g_i} < 0, \quad \frac{\partial H^i[h_1, h_2, v_1, v_2, w_i]}{\partial r_i} < 0, \; i = 1, 2$$

Our model is, therefore, described by equations (1), (5) and (8). To study its dynamic properties, we first solve for the flow (non-state) variables v_1, v_2, w_1, w_2 in terms of the state variables h_1 and h_2. Let us begin with the violence equations in (1). Write these equations as:

$$\begin{aligned} v_1(t) - V^1[h_1(t), h_2(t), v_2(t)] &\equiv F^1[v_1(t), v_2(t); h_1(t), h_2(t)] = 0 \\ v_2(t) - V^2[h_1(t), h_2(t), v_1(t)] &\equiv F^2[v_1(t), v_2(t); h_1(t), h_2(t)] = 0 \end{aligned} \tag{11}$$

The corresponding Jacobian, denoted by F, is given by:

$$F \equiv \begin{bmatrix} 1 & -\frac{\partial V^1}{\partial v_2} \\ -\frac{\partial V^2}{\partial v_1} & 1 \end{bmatrix} \tag{12}$$

For a solution to exist we must have:[19]

$$|F| = 1 - \frac{\partial V^1}{\partial v_2}\frac{\partial V^2}{\partial v_1} \neq 0$$

Assuming that this condition holds, let the solution to equations (1) be given by:

$$\begin{aligned} v_1^* &= V^{*1}(h_1, h_2) \\ v_2^* &= V^{*2}(h_1, h_2) \end{aligned} \tag{13}$$

Now, consider the effects of an increase in hate on violence. Using equations (1) we have:

$$\begin{bmatrix} 1 & -\frac{\partial V^1}{\partial v_2} \\ -\frac{\partial V^2}{\partial v_1} & 1 \end{bmatrix} \begin{bmatrix} \frac{dV^1}{dh_i} \\ \frac{dV^2}{dh_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial V^1}{\partial h_i} \\ \frac{\partial V^2}{\partial h_i} \end{bmatrix}, \; i = 1, 2$$

Thus,

$$\begin{bmatrix} \frac{dV^1}{dh_i} \\ \frac{dV^2}{dh_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial V^1}{\partial h_i} + \frac{\partial V^1}{\partial v_2}\frac{\partial V^2}{\partial h_i} \\ \frac{\partial V^2}{\partial h_i} + \frac{\partial V^2}{\partial v_1}\frac{\partial V^1}{\partial h_i} \end{bmatrix} \frac{1}{|F|} \tag{14}$$

Since $\frac{\partial V^1}{\partial h_i} > 0$, $\frac{\partial V^1}{\partial v_2} > 0$, $\frac{\partial V^2}{\partial h_i} > 0$ the effects of an increase in hate on violence depend on the sign of $|F|$. But, what is the sign of $|F|$? To answer this question, remember that we have $V^i[0, 0, 0] = 0$, $i = 1, 2$. Now,

measures (e.g., high rates of economic growth), reduce the incidence of terrorism and political violence. Honaker (2004) reports that increases in Catholic unemployment lead to increases in Republican violence and increases in Protestant unemployment lead to increases in Loyalist violence. Similarly, Santos Bravo and Mendes Dias, (2006), use 1997-2004 data for two large regions of Eurasia and find that the number of terrorist incidents is negatively associated with the level of development. See also Sørli et al., (2005), Collier and Hoeffler, (1998), Medoff (1999), Blomberg and Hess, (2002), Nafziger and Auvinen (2002).

[18] Note that it is also possible to add a "jealousy effect" in equations (8). This can be captured by taking H^i to be an increasing function of $w_j(t)$, $i \neq j$.

[19] As required by the implicit function theorem.

starting at the point where hate and violence in both countries are zero, suppose that levels of hate move away from zero in either direction. Then, if $|F| < 0$: (i) as hate increases, violence becomes negative (it becomes benevolence) and with higher levels of hate we get higher levels of benevolence, (ii) as hate decreases (it becomes love), violence become positive and with higher levels of love we get higher levels of violence. This is, clearly, unreasonable. Thus, in the following we assume that $|F| > 0$, so that.

$$\frac{\partial V^{*i}}{dh_j} > 0, \ all \ i, j = 1, 2 \tag{15}$$

Furthermore, we also have,

$$0 = V^{*1}(0, 0), \ 0 = V^{*2}(0, 0)$$

In other words, if there is no hate, there is also no violence.

Now, plugging the solution for $v_i^*, \ i = 1, 2$ into equations (4), (5) we get the solution for net growth as:

$$\begin{aligned} w_1^* &= g_1 + r^1[V^{*1}(h_1, h_2), V^{*2}(h_1, h_2)] \equiv g_1 + r^{*1}(h_1, h_2) \equiv w_1^*(h_1, h_2; g_1) \\ w_2^* &= g_2 + r^2[V^{*1}(h_1, h_2), V^{*2}(h_1, h_2)] = g_2 + r^{*2}(h_1, h_2) \equiv w_2^*(h_1, h_2; g_2) \end{aligned} \tag{16}$$

Then, from equations (5), (6) and (15) we have:

$$\frac{dw_i^*}{dh_j} < 0, \ all \ i, j = 1, 2$$

Furthermore, when $h_1 = 0, \ h_2 = 0$, we, $w_i^*(0, 0, g_i) = g_i$.

The solutions for the variables v_1, v_2, w_1, w_2 in terms of our state variables h_1 and h_2 and growth variables g_1 and g_2 can now be plugged into the two differential equations to obtain:

$$\begin{aligned} \frac{dh_1}{dt} &= H^1[h_1, h_2, V^{*1}(h_1, h_2), V^{*2}(h_1, h_2), g_1 + r^{*1}(h_1, h_2)] \equiv G^1(h_1, h_2; g_1) \\ \frac{dh_2}{dt} &= H^2[h_1, h_2, V^{*1}(h_1, h_2), V^{*2}(h_1, h_2), g_2 + r^{*2}(h_1, h_2)] \equiv G^2(h_1, h_2; g_2) \end{aligned} \tag{17}$$

The dynamic properties of the system are captured by equations (17). To understand the nature of the system we need to examine the partial derivatives of the G^i functions. Define $h \equiv (h_1, h_2)$ and $G^i_j \equiv \partial G^i(h; g_i)/\partial h_j, \ i, j = 1, 2$. Assuming that $G^i(h; g_i), \ i = 1, 2$, is continuously differentiable define the Jacobian matrix corresponding to these two equations, at $(h; g_i)$, as:

$$J \equiv \begin{bmatrix} G^1_1 & G^1_2 \\ G^2_1 & G^2_2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial H^1}{\partial h_1} + \rho^1_1 \frac{\partial V^{*1}}{\partial h_1} + \rho^1_2 \frac{\partial V^{*2}}{\partial h_1} & \frac{\partial H^1}{\partial h_2} + \rho^1_1 \frac{\partial V^{*1}}{\partial h_2} + \rho^1_2 \frac{\partial V^{*2}}{\partial h_2} \\ \frac{\partial H^2}{\partial h_1} + \rho^2_1 \frac{\partial V^{*1}}{\partial h_1} + \rho^2_2 \frac{\partial V^{*2}}{\partial h_1} & \frac{\partial H^2}{\partial h_2} + \rho^2_1 \frac{\partial V^{*1}}{\partial h_2} + \rho^2_2 \frac{\partial V^{*2}}{\partial h_2} \end{bmatrix}$$

where,

$$\rho^i_j \equiv \frac{dH^i}{dV^{*j}} = \frac{\partial H^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_i} \frac{\partial r^i}{\partial V^{*j}} \tag{18}$$

The determinant of J is given by:

$$|J| = G_1^1 G_2^2 - G_2^1 G_1^2$$

What can we say about the signs of the elements of the matrix J? Since $\frac{\partial H^i}{\partial V^{*j}} > 0$, $\frac{\partial r^i}{\partial V^{*j}} < 0$ for all i, j and $\frac{\partial H^i}{\partial w_i} < 0$, for all i, we have,[20]

$$\rho_j^i > 0, \quad i, j$$

But, since $\frac{\partial H^i}{\partial h_j} > 0$, $\frac{\partial V^{*i}}{dh_j} > 0$ for all $i \neq j$, we have,

$$G_j^i > 0, \text{ for all } i \neq j$$

Unfortunately, since the effect of a country's own hate on the evolution of its hate ($\partial H^i/\partial h_i$), is not that clear, we cannot determine the sign of G_i^i. Given that the sign of the diagonal terms are unknown this also means that so is the sign of the determinant of J. How likely is it for $\frac{\partial H^i}{\partial h_i}$ to be positive? We know that a sufficient condition for $G_i^i > 0$ is that $\frac{\partial H^i}{\partial h_i} > 0$ and a necessary and sufficient condition is that $\frac{\partial H^i}{\partial h_i} > -(\frac{\partial H^i}{\partial V^{*1}}\frac{\partial V^{*1}}{\partial h_i} + \frac{\partial H^i}{\partial V^{*2}}\frac{\partial V^{*2}}{\partial h_i} + \frac{\partial H^i}{\partial w_i^*}\frac{\partial r^{*i}}{\partial h_i})$. But, the effects of depreciation, or forgetfulness, even when they exist, may not be sufficient to outweigh the effects of violence (which consist of its direct effects and its indirect effects through its impact on the economic cost of the conflict). The case of when $G_i^i > 0$, therefore, seems quite likely.

Finally, note that while the properties of the Jacobian matrix J are related to the intrinsic styles (properties) shown in Figures 1 and 2 above, there is no a priori reason why G_j^i and H_j^i should always have the same signs (although here we have that $H_{j\neq i}^i$ and $G_{j\neq i}^i$ are both positive). The intrinsic type of a country is captured by the properties of H^i, whereas its perceived type is captured by the properties of G^i. The properties of G^i reflect both the intrinsic nature of a country and all the other factors that depend on hate and play a role in its evolution. In our example these other factors include violence and economic conditions. But in a more general model they may also include other root causes. Namely, factors that effect and are effected by hate; some of which may be the result of underlying strategic (game) considerations (e.g., by political actors). Confusion over what is intrinsic and what are root causes may explain why there so much debate and disagreement over the causes of hate, violence, extremism and terrorism.

3 Steady State

Let us first look at the special case when the two growth rates are constant. We assume that for any given (finite column) vector $h = (h_1, h_2)^T$, when g_i is "sufficiently" low we have $\frac{dh_1}{dt} > 0$ and when g_i is "sufficiently"

[20] Note that if we include a jealousy effect, then when violence increases in any single country, as a result, in *both* countries: economic costs increase, thus reducing net growth rates and hence affecting the evolution of hate. It is reasonable, however, that the impact of the reduction in a country's own growth is greater than the impact of the reduction in the rival's growth (the jealousy effect). Thus, we may have: $\rho_j^i > 0, \quad i \neq j$, if $\frac{\partial H^1}{\partial w_1}\frac{\partial r^1}{\partial V^{*1}} > -\frac{\partial H^1}{\partial w_2}\frac{\partial r^2}{\partial V^{*1}}$.

high we have $\frac{dh_1}{dt} < 0$. Given continuous $G^i(h_1, h_2; g_i)$ functions and since $\partial H^i / \partial g_i < 0$, for all h, g_i, this implies that for any given (finite) h, there exists some fixed rate of economic growth in Country i, given by g_i^h, such that,

$$\frac{dh_i}{dt} = G^i(h; g_i) \begin{cases} > 0, \text{ for all } g_i < g_i^h \\ = 0, \text{ for all } g_i = g_i^h \\ < 0, \text{ for all } g_i > g_i^h \end{cases}$$

Specifically, this implies that, in the absence of hate or violence in either country, there exists some fixed rate of economic growth in Country i, given by g_i^0, such that,

$$\frac{dh_i}{dt} = H^i[0,0,0,0,g_i] \equiv G^i(0,0;g_i) \begin{cases} > 0, \text{ for all } g_i < g_i^0 \\ = 0, \text{ for all } g_i = g_i^0 \\ < 0, \text{ for all } g_i > g_i^0 \end{cases}$$

Namely, the fixed growth rates g_1^0, g_2^0 are consistent with zero hate in both countries. These can be viewed as "normal" growth rates, in the sense that they are required in order to keep hate levels in both countries at zero levels. Clearly, in general, there is no reason why g_i^0 should be the same as the (fixed) long run growth rate (LRGR) g_i^*.[21]

Let us now examine the existence and nature of a steady state (SS) solution. First, obviously, time-dependent growth rates cannot yield a SS. Second, for fixed growth rates g_1, g_2, a SS (if it exists), is defined by the two conditions:

$$\begin{aligned} G^1(h_1, h_2; g_1) &= 0 \\ G^2(h_1, h_2; g_2) &= 0 \end{aligned} \tag{19}$$

Hence, if $g_i = g_i^0 = \textit{constant}$, $i = 1, 2$, then $h_1 = h_2 = 0$ satisfies equations (19). In other words, if growth rates in both countries are constant and equal to the normal rates, then we have a SS with zero hate. But, can we have a SS with zero hate when we do not have constant normal growth rates in both countries? Since $G^i(0,0;g_i^0) = 0$ and $\partial G^i(0,0;g_i)/\partial g_i < 0$ for all g_i (and regardless of what $g_{j \neq i}$ is), we know that there is no other fixed value of g_i that satisfies $G^i(0,0;g_i) = 0$. Hence, we can have a SS with zero hate if and only if both countries have the (fixed) normal growth. Since there is no reason why growth in both countries should be fixed and equal to the normal growth rates, if is clear that, in general, we will not have a SS with zero hate. In other words, genuine peace is not likely as a SS.

But, does a SS with *any* level of hate exist? Since the only constant growth rates are the two long run growth rates g_1^* and g_2^*, the question is whether there exists a vector $h^* \equiv (h_1^*, h_2^*)$ such that:

$$\begin{aligned} G^1(h^*; g_1^*) &= 0 \\ G^2(h^*; g_2^*) &= 0 \end{aligned} \tag{20}$$

[21] There is also no reason why, for general non-linear functions, g_i^0 should be zero for both countries.

Define the corresponding Jacobian matrix evaluated at $h^*; g^*$ as,

$$A \equiv \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

where $a_j^i \equiv G_j^i(h^*; g_i^*)$. Assuming that the determinant of the Jacobian A does not vanish at $h^*; g^*$, there exists a vector $h^* \equiv [h_1^*(g^*),\ h_2^*(g^*)]$, where $h^*(g^*)$ is a continuously differentiable and satisfies the two conditions in equations (20). Although it is obvious that

$$h_i^*(g_1^0, g_2^0) = 0,\ i = 1, 2$$

in general we expect to have $g_i^* \neq g_i^0$, so we conclude that it is not likely that we will have: $h_i^* = 0$, for $i = 1, 2$.

To be able to understand the nature of the SS we examine the properties of the two conditions (isoclines, or demarcation curves) $G^1(h; g_1^*) = 0$ (denotes as IC_1) and $G^2(h; g_2^*) = 0$ (denoted as IC_2) around the SS h^* (and given the long run growth rates g_1^*, g_2^*). Total differentiation yields the slopes of IC_1 and IC_2, at $h^* g^*$, denoted as S_1 and S_2, respectively, as:

$$S_1 \equiv \frac{dh_2}{dh_1}\Big|^1_{(h^*, g_1^*)} = -\frac{a_1^1}{a_2^1}$$

$$S_2 \equiv \frac{dh_2}{dh_1}\Big|^2_{(h^*, g_2^*)} = -\frac{a_1^2}{a_2^2}$$

While we know that $a_j^i > 0,\ for\ i \neq j$, we do not know the sign of a_i^i. Thus, we do not know the signs of S_1 and S_2 at h^*. Moreover, comparing the two slopes we get:

$$S \equiv S_1 - S_2 = -\frac{|A|}{a_2^1 a_2^2}$$

Again, since we do not the signs of a_2^2 and $|A|$, we do not know which isocline is steeper (we do not know the sign of S). It may be possible, however, to infer the likelihood of the two possible cases ($S > 0$ and $S < 0$) by examining the effects of a change in long run growth rates on the SS. A Change in g_i^* shifts IC_i, thus affecting the SS values of *both* h_1^* and h_2^*. From equations (20) we get:

$$A\frac{dh^*}{dg^*} = -A_g$$

where the 2X2 matrices $\frac{dh^*}{dg^*}$ and A_g are given by:

$$\frac{dh^*}{dg^*} = \begin{bmatrix} \frac{dh_1^*}{dg_1^*} & \frac{dh_1^*}{dg_2^*} \\ \frac{dh_2^*}{dg_1^*} & \frac{dh_2^*}{dg_2^*} \end{bmatrix}$$

$$A_g = \begin{bmatrix} a_{g_1}^1 & 0 \\ 0 & a_{g_2}^2 \end{bmatrix}$$

where $a_{g_i}^i \equiv \partial G^i(h^*; g_i^*)/\partial g_i$. Thus, the effect of a change in long term growth rates is given by:

$$\frac{dh^*}{dg^*} = -A^{-1}A_g \tag{21}$$

where A^{-1} is the inverse of A. Since $\partial G^i(h; g_i)/\partial g_i = \partial H^i/\partial w_i \, \partial w_i/\partial g_i < 0$, we also have: $a^i_{g_i} \equiv \partial G^i(h^*; g^*_i)/\partial g_i < 0$.

We know that an increase in g^*_i shifts IC_i. In fact, since the off-diagonal elements in A are positive and $a^i_{g_i} < 0$, we know that when g^*_1 increases IC_1 shifts up and when g^*_2 increases IC_2 shifts to the right. But, unfortunately, since (as was shown above) we do not know the sign of the determinant of A, or the signs of its diagonal elements, we do not know if the two isoclines are upward, or downward sloping and we do not know which is steeper. Consequently, even though we know in which direction an isocline shifts, we cannot tell what happens to the SS (the intersection of IC_1 and IC_2). Thus, in general, the effects of a change in long run growth rates are ambiguous. Nevertheless, let us examine these effects further.

From equation (21) we get:

$$\frac{dh^*_i}{dg^*_i} = -\frac{a^i_{g_i} \; a^j_j}{|A|}, \; i = 1, 2 \tag{22}$$

which is ambiguous. From equation (21) we also get:

$$\frac{dh^*_i}{dg^*_j} = \frac{a^j_{g_j} a^i_j}{|A|}, \; i \neq j, \; i, j = 1, 2 \tag{23}$$

Since $a^j_{g_j} a^i_j < 0$, we know that $sign(dh^*_i/dg^*_i) = -sign(|A|\,,\; i \neq j$. Thus, for both countries, the effects in equation (23) must have the same sign: $sign(dh^*_1/dg^*_2) = sign(dh^*_2/dg^*_1)$. There are, therefore, two possible cases: (1) both dh^*_1/dg^*_2 and dh^*_2/dg^*_1 are negative, (2) both dh^*_1/dg^*_2 and dh^*_2/dg^*_1 are positive.

Although it may be obvious, it is still useful to note that (assuming that $a^i_{g_i}$ and a^i_j, $i, j = 1, 2$, are all non-zero) a change in the LRGR in *one* country affects the SS hate levels in *both* countries: this is simply a reflection of the interaction of the effects on the two isoclines. This, however, introduces two important elements into the analysis. First, even though g^*_j does not appear in country $i's$ motion equation, a higher LRGR in country j will decrease, or increase the SS level of hate in country $i \neq j$. In other words, "indirect" elements of envy/compassion, are introduced into the model by the interaction of the two isoclines. Second, even though, by assumption, g^*_i has a negative effect on a country's *evolution* of hate ($a^i_{g_i} < 0$), an increase in g^*_i may either decrease, or *increase* its *own* SS level of hate. In other words, the SS equilibrium response to an increase in its growth rate may be inconsistent with its "inherent" response (as captured by $a^i_{g_i}$; the response of its evolution of hate).

Altogether there are six possible cases, depending on the properties of the A matrix (which determine the slopes, S_i, and difference in slopes, S, of the isoclines). These cases are summarized in Table 3 below.[22]

[22] Note that when a^1_1 and a^2_2 have opposite signs, then $|A| < 0$, thus there are no entries in the last two columns of the first row in Table 3.

$\lvert A\rvert \backslash S_i$	$a_i^i < 0 \rightarrow S_i > 0$	$a_i^i > 0 \rightarrow S_i < 0$	$a_1^1 > 0,\ a_2^2 < 0$	$a_1^1 < 0,\ a_2^2 > 0$
			$\rightarrow S_1 < 0, S_2 > 0$	$\rightarrow S_1 > 0, S_2 < 0$
$\lvert A\rvert > 0$	Case (i) $\frac{dh_1^*}{dg_1^*} < 0$ $\frac{dh_i^*}{dg_j^*} < 0,\ S > 0$	case (iv) $\frac{dh_1^*}{dg_1^*} > 0$ $\frac{dh_i^*}{dg_j^*} < 0,\ S < 0$	$----------$	$----------$
$\lvert A\rvert < 0$	Case (iii) $\frac{dh_1^*}{dg_1^*} > 0$ $\frac{dh_i^*}{dg_j^*} > 0,\ S < 0$	Case (iia) $\frac{dh_1^*}{dg_1^*} < 0$ $\frac{dh_i^*}{dg_j^*} > 0,\ S > 0$	Case (iib) $\frac{dh_1^*}{dg_1^*} > 0, \frac{dh_2^*}{dg_2^*} < 0$ $\frac{dh_i^*}{dg_j^*} > 0,\ S < 0$	Case (iic) $\frac{dh_1^*}{dg_1^*} < 0, \frac{dh_2^*}{dg_2^*} > 0$ $\frac{dh_i^*}{dg_j^*} > 0,\ S > 0$

Table 3: Properties of A

Although there are six possible cases, not all seem equally "reasonable". Specifically, it seems reasonable that a country's SS level of hate should decrease when its own LRGR increases. This should be true at least for one of the two countries (namely we should have $dh_i^*/dg_i^* < 0,$ for at least one country). It is, of course, possible that this may not be true, but it would indeed seem unreasonable if for both countries h_i^* should increase with g_i^*. If this requirement is true for both countries, all but cases (i) and (iia) would be eliminated. If it is true for at least one country, cases (iii) and (iv) would be eliminated. What about the "cross effect" of a change in g_i^* on h_j^*? Do we expect $dh_i^*/dg_j^*,\ i \neq j$ to be positive or negative? If $dh_i^*/dg_j^* > 0$ we have what we referred to above as indirect envy/compassion (when g_j^* increases/decreases h_i^* increases/decreases). Although the envy part does not "sound appealing", altogether this does not seem unreasonable. What about the other case where $dh_i^*/dg_j^* < 0$? In this case, a country's SS level of hate will be low when its rival's growth rate is high and it will be high when its rival's growth rate is low. Regardless of ethical considerations, this seems less "reasonable". If, in addition to eliminating cases without $dh_i^*/dg_i^* < 0,$ for at least one country, we were to eliminate also the cases where $dh_i^*/dg_j^* < 0$, we would end up with three cases only: cases (iia), (iib) and (iic).

Let us consider the implications of cases (i)-(iic) above regarding the nature of the SS. Specifically, we are interested in finding out what the likely SS solutions are and, in particular, whether they are likely to involve no hate. First, as was pointed out above, in general, there is no reason why we should have $g_i^* \neq g_i^0$, so it is not likely that we will have zero hate in both countries. Second, whether the SS levels of hate are positive, or negative depends on (i) whether the long run growth rates are higher or lower than the normal rates (ii) which of the six cases above occurs. This is summarized in Table 4 below (an entry/pair $+-$ means that $h_1^* > 0$ and $h_2^* < 0$; more than one entry means that, at least for one country, the sign is ambiguous - it depends on $g_1^* - g_1^0$, relative to $g_2^* - g_2^0$). As Table 4 shows, a given configuration $(g_1^*,\ g_2^*)$ and (g_1^0, g_2^0) does not necessarily tell us if h_1^* and h_2^* are positive or negative. Specifically, just because $g_1^* > g_1^0$ it does not mean that $h_1^* < 0$ (e.g., in *case* (iib) we have $h_1^* > 0$). Furthermore, in general, a SS involves non-zero levels of hate (love). Table 4 also shows that, in general, a SS involves non-zero levels of hate (love).

	$g_1^* > g_1^0,\ g_2^* = g_2^0$	$g_1^* > g_1^0,\ g_2^* > g_2^0$	$g_1^* < g_1^0,\ g_2^* = g_2^0$	$g_1^* < g_1^0,\ g_2^* < g_2^0$	$g_1^* > g_1^0,\ g_2^* < g_2^0$
case (i)	$--$	$--$	$++$	$++$	$-+/--/++$
case (iia)	$-+$	$-+/+-/++$	$+-$	$-+/+-/--$	$-+/+-/++$
case (iib)	$++$	$++/+-$	$--$	$--/-+$	$-+/++$
case (iic)	$-+$	$-+/++$	$+-$	$+-/--$	$-+/--$

Table 4: The Signs of the SS levels of hate

4 Solution of the Differential Equations

To solve the two differential equation, we linearize the equations $G^i(h_1, h_2, g_i)$ at the SS point (h^*, g^*). The linear approximation of the two hate motion equations is then:

$$\begin{aligned} G^1(h,g) &\simeq q^1 + a_1^1 h_1 + a_2^1 h_2 + a_{g_1}^1 g_1 \\ G^2(h,g) &\simeq q^2 + a_1^2 h_1 + a_2^2 h_2 + a_{g_2}^2 g_2 \end{aligned}$$

where $q^i \equiv -\ ({a^i}_1 h_1^* + {a^i}_2 h_2^* + a_{g_i}^i g_i^*)$,[23] so that the (linearized) differential equations can be written as the non-autonomous system (NAS),

$$\frac{dh}{dt} = Ah + \eta \tag{24}$$

where $\frac{dh}{dt}$ and η are the 1X2 column vectors $(\frac{dh_1}{dt}, \frac{dh_2}{dt})^T$ and $(q^1 + a_{g_1}^1 g_1, q^2 + a_{g_2}^2 g_2)^T$, respectively.

First we find the solution to the homogeneous system (HS) $\frac{dh}{dt} = Ah$. Let the vectors $h^1 = \begin{bmatrix} h_1^1 \\ h_2^1 \end{bmatrix}$ and $h^2 = \begin{bmatrix} h_1^2 \\ h_2^2 \end{bmatrix}$ be two linearly independent solutions to the HS (note that h^i with a superscript stand for the a solution vector, whereas h_i with a subscript stands for the state variable -level of hate- in country i). Then, $h^m = \begin{bmatrix} h_1^m \\ h_2^m \end{bmatrix}$, where $h_i^m = c_1 h_i^1 + c_2 h_i^2$, is the general solution to the HS.

The two corresponding eigenvectors, defined as $z^1 = \begin{bmatrix} z_1^1 \\ z_2^1 \end{bmatrix}$ and $z^2 = \begin{bmatrix} z_1^2 \\ z_2^2 \end{bmatrix}$, are obtained from the following equations:

$$\begin{aligned} Az^1 &= \lambda_1 z^1 \\ Az^2 &= \lambda_2 z^2 \end{aligned}$$

Hence, the general solution to the HS is:

$$h^m = \begin{bmatrix} h_1^m \\ h_2^m \end{bmatrix} = \begin{bmatrix} c_1 z_1^1 e^{\lambda_1 t} + c_2 z_1^2 e^{\lambda_2 t} \\ c_1 z_2^1 e^{\lambda_1 t} + c_2 z_2^2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} z_1^1 e^{\lambda_1 t} & z_1^2 e^{\lambda_2 t} \\ z_2^1 e^{\lambda_1 t} & z_2^2 e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \equiv \Psi c$$

If the determinant of Ψ is non-zero, we refer to it as the fundamental matrix corresponding to the HS above. We assume that initial time is $t_0 = 0$ and the vector of initial values of hate, $h(0)$, is given by,

$$h(0) \equiv [h_1(0), h_2(0)] = [h_1^0, h_2^0] \equiv h^0$$

[23] And, of course, $G^i(h^*; g_i^*) = 0$.

Given the fundamental matrix and using the initial conditions, we can obtain the values of the elements in the vector c by solving the equations $h^0 = \Psi(0)c$ to obtain:

$$c = \Psi^{-1}(0)\ h^0$$

The solution to the HS is, therefore, given by

$$h^m = \Psi(t)\Psi^{-1}(0)\ h^0$$

Now, define the (transition) matrix $\Phi(t,\tau)$ as:

$$\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$$

then, the solution to the HS is given by:

$$h^m = \Phi(t,0)\ h^0$$

The general solution to the NAS in (24), defined as h^s, is given by the sum of the solutions to the HS (h^m) and the particular solution to the NAS, denoted as h^p :

$$h^s = h^m + h^p$$

where h^p is given by:

$$\begin{aligned} h^p &= \Psi(t)\int_0^t \Psi^{-1}(\tau)\eta(\tau)d\tau \\ &= \int_0^t \Phi(t,\tau)\eta(\tau)d\tau \end{aligned}$$

Hence, the solution to the non-autonomous is given by (the variation of parameter formula):

$$\begin{aligned} h^s(t) &= \Psi(t)\Psi^{-1}(0)\ h^0 + \Psi(t)\int_0^t \Psi^{-1}(\tau)\eta(\tau)d\tau = \\ & \Phi(t,0)\ h^0 + \int_0^t \Phi(t,\tau)\eta(\tau)d\tau \end{aligned}$$

5 Stability

We begin by examining the HS. The characteristic equation corresponding to the HS is given by,

$$\lambda^2 - (a_1^1 + a_2^2)\lambda + |A| = 0$$

and its characteristic roots are:

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{1}{2}[(a_1^1 + a_2^2 \pm \sqrt{(a_1^1 + a_2^2)^2 - 4\ \ |A|}] \\ &= \frac{1}{2}[(a_1^1 + a_2^2) \pm \sqrt{\Delta}] \end{aligned}$$

where $\Delta \equiv (a_1^1 + a_2^2)^2 - 4\ |A|$ is the corresponding discriminant. But, since $a_{ij} \equiv \partial G^i(h^*, g_i^*)/\partial h_j > 0,\ i \neq j,$ $i, j = 1, 2$, we have:

$$\Delta \equiv (a_1^1 + a_2^2)^2 - 4\ |A| = (a_1^1 - a_2^2)^2 + 4a_2^1 a_1^2 > 0$$

Thus, we conclude that the solution involves two distinct real roots. Consequently, a cyclical pattern of hate in the HS is not possible. Moreover, given that cyclical patterns of hate cannot occur in the HS and given the monotonic relationship between violence and hate (in equation (15) we have $\partial V^{*i} \partial V^{*i}/\partial h_j > 0,\ all\ i, j = 1, 2$), it also follows that the HS itself cannot give rise to cyclical patterns of violence. Similarly, this also implies that, *for fixed values* of *gross* rates of economic growth, we cannot have cyclical patterns of *net* rates of economic growth $(w_1^*(h_1, h_2; g_1)$. Cyclical patterns of hate, violence and net economic growth can, therefore, occur only due to possible cyclical patterns in the non-autonomous component, $\eta_i(t) = q^i + a_{g_i}^i g_i(t)$, which in turn reflect cyclical patterns in $g_i(t)$.

The only question is, therefore, whether the SS equilibrium in the HS is stable, or unstable. In principle, the following cases are possible in the HS: (I) if both roots are negative we have a stable node (II) if both roots are positive we have an unstable node and (III) if the roots have opposite signs we have a saddle point.

Let us, therefore, look at the six possible cases above. In the three "most reasonable" cases ((iia), (iib) and (iic)) we have $|A| < 0$. Since

$$\lambda_1 \lambda_2 = |A| < 0$$

λ_1 and λ_2 must have opposite signs. Hence, we conclude that in all three "most reasonable" cases the solution to the HS is a saddle-point.

In cases (i), (iii) and (iv) that were deemed to be "less reasonable" we have the following stability properties. In case (i) we have $|A| > 0$, so λ_1 and λ_2 must have the same sign. But, since $a_j^j < 0$ we have $\lambda_1 + \lambda_2 = a_1^1 + a_2^2 < 0$, so λ_1 and λ_2 must both be negative, hence the solution to the HS is a stable node. In case (iii) we have $a_i^i < 0$, and $|A| < 0$, so we have a saddle-point. In case (iv) we have $|A| > 0$ and $a_i^i > 0$, so λ_1 and λ_2 must be both positive, so we have an unstable node.

Since in all three reasonable cases all have a SP, we conclude that the most likely outcome is, in fact, a SP. Namely, the HS is most likely to be unstable. Can we refine this even further? Remember that in order to have $a_i^i < 0$, we require that (at $h^*; g_i^*$): $\frac{\partial H^i}{\partial h_i} < -(\frac{\partial H^i}{\partial V^{*1}} \frac{\partial V^{*1}}{\partial h_i} + \frac{\partial H^i}{\partial V^{*2}} \frac{\partial V^{*2}}{\partial h_i} + \frac{\partial H^i}{\partial w_i^*} \frac{\partial r^{*i}}{\partial h_i})$; namely, the effects of depreciation, or forgetfulness, if they exist, must outweigh the effects of persistence plus the direct and indirect effects of violence. Since this is likely not to occur, cases (iib) and (iic) are not likely to occur. The most likely case is, therefore, case (iia) which yields a saddle point. We should remember, however, that hate neither conforms to, nor is based on, reason. Asking whether a particular case is reasonable or not may, in itself, not be desirable or even reasonable. Hence, it is perhaps best not to exclude cases that seem unreasonable.

We showed above that since $a_{12} > 0$ and $a_{21} > 0$, cyclical patterns of hate cannot occur in the HS. Let us now briefly comment on the circumstances (assumptions) under which cycles may actually occur in the HS. Suppose that we introduce direct jealousy effects. Specifically, suppose that the evolution of hate in each country depends on economic conditions in *both* countries, so that the differential equations are given by:

$$\frac{dh_i}{dt} = H^i(h_1, h_2, v_1, v_2, w_1, w_2),\ i = 1, 2 \tag{25}$$

A jealousy effect in country i occurs if $\frac{\partial H^i}{\partial w_j} > 0$, $i \neq j$. Given equation (25) we now have to re-write equation (18) above as:

$$\rho^i_j \equiv \frac{dH^i}{dV^{*j}} = \frac{\partial H^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_i}\frac{\partial r^i}{\partial V^{*j}} + \frac{\partial H^i}{\partial w_j}\frac{\partial r^j}{\partial V^{*j}},\ i \neq j$$

Thus, for example, if country i has a jealousy effect, we may not have $\rho^i_j > 0$, $i \neq j$ and consequently, we may (but not necessarily) have $a^i_j \equiv G^i_j(h^*; g^*_i) < 0$, $i \neq j$. We can define jealousy in country i to be "sufficiently strong" if it yields $a^i_j < 0$, $i \neq j$. Hence, if both countries have sufficiently strong jealousy ($a^1_2 < 0$ and $a^2_1 < 0$), we still have $\Delta = (a^1_1 - a^2_2)^2 + 4a^1_2 a^2_1 > 0$, so no cycles are possible. But, if only one country has sufficiently strong jealousy, whereas the other country does not (it either has no jealousy, or jealousy which is not sufficiently strong), then since a^1_2 and a^2_1 have opposite signs, we may have $\Delta = (\gamma^1_1 + \gamma^2_2)^2 + 4\gamma^1_2\gamma^2_1 < 0$ and consequently, cycles may occur. Sufficiently strong jealousy in one and only one country is, therefore, a necessary condition for cycles in the HS.

Note that if the evolution of hate in country i is negatively affected by the level of hate in country $j \neq i$ (i.e., we have $\partial H^i/\partial h_j < 0$) we may also get $a^i_j < 0$, $i \neq j$, possibly leading to cycles in the HS. This, however, is not very reasonable. Moreover, even if the evolution of hate in country i is negatively affected by the level of hate in country j, this would still need to be true for one and only one country, so that a^1_2 and a^2_1 would have opposite signs.

Finally, since in the most likely cases the HS is unstable, so is the corresponding NAS. At the same time, given growth rates that converge to fixed (finite) long run rates, it follows that, in the less likely case, when the HS is stable (case(i)), so is the NAS.

6 Example: Linear Model

In this section we provide an example of a linear model that enables us to examine the effects of changes in various parameters on the nature and stability of the equilibrium. We assume that equations (1) are given by the following linear equations:

$$\begin{aligned} v_1 &= s^1_1 h_1 + s^1_2 h_2 + d_1 v_2 \\ v_2 &= s^2_1 h_1 + s^2_2 h_2 + d_2 v_1 \end{aligned} \tag{26}$$

or,

$$\begin{bmatrix} 1 & -d_1 \\ -d_2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s^1_1 & s^1_2 \\ s^2_1 & s^2_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{27}$$

where $s^i_j > 0$ and $d_i > 0$ for all $i, j = 1, 2$ and the 2X2 matrix on the left hand side corresponds to the Jacobian matrix F in equation (12). As we noted above, $|F| > 0$, thus we have:

$$\phi \equiv 1/(1 - d_1 d_2) > 0.$$

Define the 2X2 matrix on the right hand side of equation (27) as S and the column vectors of hate and violence as h and v, respectively. The solution for violence is, therefore, given by:

$$v = \phi S h \tag{28}$$

Next, we assume that $w_i[v_1(t), v_2(t), t]$ in equation (5) is given by:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} - \begin{bmatrix} k^1_1 & k^1_2 \\ k^2_1 & k^2_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{29}$$

where $k^i_j > 0,$ for all $i, j = 1, 2$. Define the 2X2 matrix on the right hand side of equation (29) as K and the column vectors of net and gross growth rates w and g, respectively. The solution for violence is, therefore, given by,

$$w = g - Kv$$

But, since we are using a linear model in this example and growth rates are, in general, not linear, we need to do one of two things. First, can simply take the growth rates to be fixed and given by their long run values of g^*_1 and g^*_2. Second, we can linearize g_i at its long run value, which again gives us the same result. Namely, the linearized value of g_i, at its long run value, is simply the long run values of g^*_i. Thus, we write the solution for w as:

$$w = g^* - Kv$$

Finally, we assume that equations (8) above can be written as:

$$\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & \alpha^2_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \theta^1_1 & 0 \\ 0 & \theta^2_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{30}$$

where: (i) $\alpha^i_j > 0,$ for all $i \neq j$, but α^i_i, $i = 1, 2$, may be positive, negative, or zero (ii) $b^i_j > 0$ for all $i, j = 1, 2$ and $\theta^i_i > 0$ for all $i = 1, 2$. Defining the three 2X2 matrices in equation (30) as A, B and θ, respectively, and defining the 1X2 column vector on the left hand side as $\frac{dh}{dt}$, the equations can be written as:

$$\frac{dh}{dt} = Ah + Bv - \theta w$$

Substituting the solution for violence and growth from equations (28) and (29) our system of two differential equations can be written as:

$$\frac{dh}{dt} = [A + \phi(B + \theta K)S]h - \theta g^*$$

or,

$$\frac{dh}{dt} = \gamma h - \theta g^* \tag{31}$$

where the 2X2 matrix γ is defined by:

$$\gamma \equiv \alpha + \phi(B + \theta K)S \tag{32}$$

and $\gamma_j^i > 0$, for all $i \neq j$, but γ_i^i, $i = 1, 2$, may be positive, negative, or zero.

First, note that in order to have a SS with zero hate in the linear model (assuming that $|\gamma| \neq 0$) we must have zero growth in both countries. There is, of course, no reason why this should be the case. In fact, from equation (31), the SS level of hate is now given by:

$$h^* = \gamma^{-1}\theta g^*$$

which will not be zero in either country unless both long run growth rates are zero.

Let us now examine the dynamic properties and stability of the system. The characteristic equation corresponding to equation (31) is given by,

$$\lambda^2 - (\gamma_1^1 + \gamma_2^2)\lambda + |\gamma| = 0$$

and its characteristic roots are:

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{1}{2}[(\gamma_1^1 + \gamma_2^2) \pm \sqrt{(\gamma_1^1 + \gamma_2^2)^2 - 4\ |\gamma|}] \\ &= \frac{1}{2}[(\gamma_1^1 + \gamma_2^2) \pm \sqrt{\Delta}] \end{aligned} \tag{33}$$

where $\Delta \equiv (\gamma_1^1 + \gamma_2^2)^2 - 4\ |\gamma|$ is the corresponding discriminant. But, since $\gamma_{ij} > 0$, for $i \neq j$, we have:

$$\Delta \equiv (\gamma_1^1 + \gamma_2^2)^2 - 4\ |\gamma| = (\gamma_1^1 - \gamma_2^2)^2 + 4\gamma_2^1\gamma_1^2 > 0$$

Thus, the solution involves two real roots. Consequently, a cyclical pattern of hate is not possible. The only question is whether the SS equilibrium is stable, or unstable. The following cases are possible in this model: (i) if both roots are negative we have a stable node (ii) if both roots are positive we have an unstable node and (iii) if the roots have opposite signs we have a saddle point. Under what conditions are these three possibilities likely to occur? In the previous section we discussed what the "reasonable" cases may be. In this section, with a linear example, we can go a bit further and examine the effects of various parameters on the likelihood of different stability configurations.

First, consider the effects of an increase in γ_{ii}. From equation (33) we obtain:[24]

$$\frac{\partial \lambda_1}{\partial \gamma_i^i} = \frac{1}{2\sqrt{\Delta}}(\sqrt{\Delta} + (\gamma_i^i - \gamma_j^j)) \geq 0, \ \ i \neq j \tag{34}$$

[24] Since with $\gamma_2^1\gamma_1^2 > 0$, we have: $\sqrt{\Delta} + (\gamma_i^i - \gamma_j^j) = \sqrt{(\gamma_j^j - \gamma_i^i)^2 + 4\gamma_2^1\gamma_1^2} + (\gamma_i^i - \gamma_j^j) > \sqrt{(\gamma_j^j - \gamma_i^i)^2} + (\gamma_i^i - \gamma_j^j) = 0.$

$$\frac{\partial \lambda_2}{\partial \gamma_i^i} = \frac{1}{2\sqrt{\Delta}}(\sqrt{\Delta} + (\gamma_j^j - \gamma_i^i)) \geq 0, \;\; i \neq j$$

where $\Delta > 0$.

Thus, as γ_i^i increases, both characteristic roots increase, so it becomes more likely to have an unstable solution. But, since γ_i^i is a complicated expression that involves many of the parameters, it would be useful to identify what will cause it to increase. From equations equation (32) we have:

$$\gamma_i^i = \phi(\theta_i^i[k_1^i(s_i^1 \; + d_1 s_i^2) \; + k_2^i(s_i^2 + d_2 s_i^1)] - (1 - d_2 d_1)\alpha_i^i$$
$$+(b_1^i d_1 + b_2^i)s_i^2 + b_2^i d_2 s_i^1 + b_1^i s_i^1, \;\; i = 1, 2 \tag{35}$$

where $\phi \equiv 1/(1 - d_1 d_2) > 0$. Since all the parameters (were define so that they) are positive, it follows that an increase in any of the parameters on the right hand side of equation (35), except for α_i^i, will increase γ_i^i; thus making instability more likely.[25] On the other hand, an increase α_i^i will decrease γ_i^i, thus making stability more likely.

Next, consider the effects of an increase in γ_j^i , $i \neq j$. From equation (33) we obtain:[26]

$$\frac{\partial \lambda_1}{\partial \gamma_j^i} = \frac{\gamma_i^j}{\sqrt{m}} > 0, \;\; i \neq j$$

$$\frac{\partial \lambda_2}{\partial \gamma_j^i} = \frac{1}{2\sqrt{m}}(\sqrt{m} + (\gamma_j^j - \gamma_i^i)) \geq 0, \;\; i \neq j$$

Thus, as γ_j^i increases, both characteristic roots increase, so it becomes more likely to have an unstable solution. From equations equation (32) we have:

$$\gamma_j^i = \phi[(\theta_i^i k_i^i + b_i^i)(s_j^i + d_i s_j^j) + (1 - d_2 d_1)\alpha_j^i + b_j^i(d_j s_j^i + s_j^j) + \theta_i^i k_j^i(s_j^j + d_j s_j^i)] \tag{36}$$

Thus, an increase in any of the parameters on the right hand side of equation (36), will increase the corresponding γ_j^i, thus make instability more likely.[27]

Thus, for example, the above results imply that instability becomes more likely if there is an increase in: (i) responsiveness to economic condition (captured by θ_i^i), or violence (captured by b_j^i), (ii) vengefulness (unwillingness to forgive, as captured by α_j^i) and (iii) marginal cost of violence (captured by k_j^i). On the other hand an increase in forgetfulness (depreciation - captured by α_i^i) makes stability more likely.

7 Conclusion

This paper provides a simple model that explains the interdependence between hate, violence and economic conditions and their dynamic properties. We define the ideal state of genuine peace and show that, in general,

[25] Note that $\frac{\partial \gamma_i^i}{\partial d_1} = \frac{1}{(1-d_2 d_1)^2}$ $(b_2^1 s_1^1 + \theta_1^1 k_2^1 s_1^1 + d_1 \theta_1^1 k_1^1 s_1^1 + b_1^1 d_1^2 s_1^2 + d_1 \theta_1^1 k_2^1 s_1^2 + d_1 b_1^1 s_1^1 + \theta_1^1 k_1^1 d_1^2 s_1^2 + d_1 b_2^1 s_1^2) > 0$.

[26] The sign of the second effect is obtained in the same way as in equation (34).

[27] Note that from the previous footnote it can be seen that $\frac{\partial \gamma_j^i}{\partial d_1} > 0$.

such a state is not likely to be an equilibrium. Although we show that a time-dependent economic growth process that effects the evolution of hate (and converges to some long run value) can yield a long run steady state, this steady state will not be one with zero hate and zero violence. Moreover, we show that a better long run economic environment does not necessarily result in lower equilibrium levels of hate and violence. We examine the dynamic properties of the hate and violence and show that, under reasonable conditions, cycles of hate and violence cannot occur. This implies that the dynamics of hate and violence in itself cannot result in cyclical patterns of (net) economic well-being. Specifically, cyclical patterns of (net) economic well-being can occur only due to cyclical patterns of gross growth rates. We also show that while it is possible to have either stable, or unstable equilibria, the most likely equilibrium is unstable (a saddle point). Using a linear example, we show that instability becomes more likely if there is an increase in the responsiveness to economic condition and violence; the unwillingness to forgive; the marginal cost of violence and the "length" of a country's memory.

8 References

Appelbaum, E. and E. Katz, (2007), "Political Extremism in the Presence of a Free Rider Problem", Public Choice 133, 1-2, 31-40.

Appelbaum, E., (2008), "Extremism as a strategic tool in conflicts", Journal of Economic Behavior & Organization 68, 2, 352-364.

Atkinson, S. E., Sandler, T. and Tschirhart, J., (1987), "Terrorism in a bargaining framework", Journal of Law and Economics, 30, 1, 1-21.

Bandarage, A., (2004), "Beyond Globalization and Ethno-Religious Fundamentalism", Development, 47, 1, 35-41.

Blomberg, S. B., Hess, G. D. and Weerapana, A., (2004), "Economic Conditions and Terrorism", European Journal of Political Economy, 20, 2, 463-478.

Blomberg, B.S. and G.D. Hess, (2002), "The temporal links between conflict and economic activity", Journal of Conflict Resolution, 46, 74-90.

Cameron, S., (2009), The Economics of Hate, Edward Elgar Publishing.

Breton, A., et al., eds., (2002), Political Extremism and Rationality, Cambridge University Press.

Collier, P. and A. Hoeffler, (1998), "On economic causes of civil war", Oxford Economic Papers, 50, 4, 563-73.

DeAngelis, D.L., W. Post and C. Travis, (1985), Positive Feedback in natural Systems, Springer-Verlag.

Eizenstat, S., J. E. Porter and J. Weinstein, (2005), "Rebuilding Weak States", Foreign Affairs, January/February.

Enders, W. and T. Sandler, (2006), The political economy of terrorism, Cambridge university press.

Esteban, J. and D. Ray, (1999), "Conflict and Distribution", Journal of Economic Theory, 87, 2, 379-415.

Herman, E.S. and N. Chomsky, (1988), Manufacturing consent: The Political Economy of the Mass Media, Pantheon Books.

Festinger,L. and J.M. Carlsmith, (1959), "Cognitive Consequences of Forced Compliance", Journal of Abnormal and Social Psychology, 58, 203-210.

Glaeser, E.L., (2005), "The Political Economy of Hatred", The Quarterly Journal of Economics, 120, 1, 45-86.

Hirsch, M.W. and H. Smith, (2006), "Chapter 4 Monotone Dynamical Systems", Handbook of Differential Equations: Ordinary Differential Equations, 2, 239–357.

Honaker, J., (2004), "Unemployment and Violence in Northern Ireland: A Missing Data Model for Ecological Inference.", Presented at the Annual Meetings of the Midwest Political Science Association.

Levin, J. and G. Rabrenovic, (2001), "Hate crimes and ethnic conflict", American Behavioral Scientist, 45, 4, 574-587.

Levy, A. (2008), "From Hatred to Genuine Peace: A Dynamic Formulation of Conflict Resolution", International Mathematical Forum, 3, 1-4, 43-47.

Medoff, H., (1999), "Allocation of Time and Hateful Behavior: A Theoretical and Positive Analysis of Hate and Hate Crimes", American Journal of Economics and Sociology, 58, 4, 959-973.

Minow, M., (2002), Breaking the Cycles of Hatred: Memory, Law, and Repair, Princeton University Press.

Muller, E.N. and E. Weede, (1990), "Cross-National Variation in Political Violence: A Rational Action Approach", The Journal of Conflict Resolution, 34, 4, 624-651.

Nafziger, E.W. and J. Auvinen, (2002), "Economic development, inequality, war and state violence", World Development, 30, 153-163.

Nozick, R. ed., (1997), "The Characteristic Features of Extremism", in Socratic Puzzles, Cambridge: Harvard University Press, 296-299.

Pape, R.A., (2003), "The Strategic Logic of Suicide Terrorism", The American Political Science Review, 97, 3, 343-361.

Piazza, J,A., (2006), "Rooted in Poverty?: Terrorism, Poor Economic Development, and Social Cleavages", Terrorism and Political Violence, 18, 1, 159 - 177.

Reynal-Querol, M. (2002), "Ethnicity, political systems, and civil wars", Journal of Conflict Resolution, 46, 29-54.

Rinaldi, S., (1998), "Laura and Petrarch: An Intriguing Case of Cyclical Love Dynamics", SIAM Journal on Applied Mathematics, 58, 4, 1205-1221.

Sandler, T. and Enders, W., (2004), “An Economic Perspective on Transnational Terrorism”, European Journal of Political Economy, 20, 2, 301-316.

Santos Bravo, A., B. and C. M. Mendes Dias, (2006), “Depravation, Islamism and Geopolitical Factors”, Defence and Peace Economics, 17, 4, 329 - 341.

Schelling, T.C., (1960), The Strategy of Conflict , Harvard University Press.

Sherif, M. (1966), In Common Predicament: Social Psychology of Intergroup Conflict and Cooperation, Boston: Houghton Mifflin Company, 24–61.

Sørli, M. E., N.P. Gleditsch and H. Strand, (2005), “Why Is There so Much Conflict in the Middle East?”, The Journal of Conflict Resolution, 49, 1, 141-165.

Sprott, J. C., (2004), “Dynamical Models of Love”, Nonlinear Dynamics, Psychology, and Life Sciences, 8, 3, 303-313.

Sternberg, R.J., and K. Sternberg, (2008), The Nature of Hate, Cambridge University Press.

Strogatz, S. H. (1994). Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering. Reading, MA: Addison- Wesley.

Wintrobe, R. (2006). Rational extremism: the political economy of radicalism. Cambridge University Press.

Highlights

- Hate, violence and economic well-being are dynamically interdependent.
- Genuine peace is unlikely to be an equilibrium.
- Improved well-being does not necessarily reduce long-run hate and violence.
- The most likely equilibria are unstable and non-cyclical.
- Instability is more likely with greater unwillingness to forgive and forget.