An Application of Duality Under Uncertainty

Elie Appelbaum *
York University

October 1994, Revised February 1996

Abstract

The purpose of this paper is to provide an empirical application of duality under uncertainty. Using the indirect utility function we develop a simple, empirically implementable framework, that can be used to estimate the effects of price uncertainty on firms' behaviour. The model is used to test implications of the theory and to identify the effects of uncertainty on input demand.

In an empirical example we estimate the effects of (energy and output) price uncertainty on input demand by the U.S. manufacturing industry. We find that production responses indicate the existence of risk aversion and are consistent with behaviour under decreasing absolute risk aversion. The actual effects of uncertain prices on input demand were, however, rather small.

*I would like to thank L. Epstein and A. Ullah for their helpful comments. Financial support from SSHRC and York University is gratefully acknowledged. An earlier version of this paper was presented at the European Economic Association Meetings in Helsinki 1993.

1 Introduction

The effects of uncertainty on firm behaviour have been the subjects of numerous theoretical studies in the literature. Recently, applications of duality in production theory have been extended choice under uncertainty[1]. It is surprising, however, that thus far, with very few exceptions, there are very few empirical applications which examine the effects of uncertainty on firms' decisions[2].

The purpose of this paper is apply these theoretical models and provide an example of an empirical application of duality under uncertainty. Applying duality under uncertainty, we provide a simple, empirically implementable framework for the analysis of firms' behaviour under uncertainty. Using the induced (indirect) utility function, we develop a model which can be used to capture the effects of (input and output) price uncertainty on firms' behaviour. The model is used to test implications of the theory and to identify the effects of uncertainty on input demand. In an empirical example we estimate the effect of (energy and output) price uncertainty on input demand by the U.S. manufacturing industry. We find that production responses indicate the existence of risk aversion and are consistent with behaviour under decreasing absolute risk aversion.

2 Theoretical Framework

2.1 The Indirect Utility Functional

Consider a firm that uses a vector of n inputs, $x \in \mathcal{B}$, a compact subset in $\mathcal{R}^n_+$, to produce an output, $y \in \mathcal{C}$, a compact subset in $\mathcal{R}^1_+$, given its technology, which is summarized by the production possibilities set $\mathcal{T}$. We make the standard assumptions regarding the production possibilities set, namely, $\mathcal{T}$ is taken to be non-empty, monotonic, convex and closed. The firm makes its decisions facing uncertain output and inputs prices. We assume that input and output prices are given by the positive and bounded random variables $p \in [0, I_p]$ and $w_i \in [0, I_i],\ \ i = 1...n$. The random variables p, w are distributed according to the distribution function $G'(p, w)$, with the support $\mathcal{A}' \equiv \{(p, w) : p \in [0, I_p],\ w_i \in [0, I_i]\}$.

We assume that the firm is competitive in input markets, but we do not make specific

[1] See for example Pope (1980), Machina (1984), Chavas (1985), Chavas and Pope (1985), Dalal (1990).

[2] For examples of theoretical studies see: Sandmo (1971), Batra and Ullah (1974), Hartman (1976), Epstein (1978), (1980), Pope (1980), Chavas (1985), Chavas and Pope (1985), Appelbaum and Katz (1986), Dalal (1990). Epstein (1980) examines possible functional forms for restricted profit functions, to be used in a two stage problem, where some choices are made after uncertainty has been resolved. His analysis focuses, however, on the properties of the restricted profit function as a function of known prices. For references of empirical studies under certaity see Fuss and McFadden (1978), Jorgenson (1986). For examples of empirical studies that consider the effects of uncertainty on firm behaviour see Parkin (1970), Just (1974), Antonovitz and Roe (1986), Appelbaum (1991), Appelbaum and Kohli (1995).

assumptions regarding its output market structure. The firm's revenues are given by $R = py$, where the random variables (R, w), are distributed according to the distribution function $G(R, w)$, with the support $\mathcal{A} \equiv \{(R, w) : R \in [0, I_R],\ w_i \in [0, I_i]\}$, where $I_R = yI_p$. It is useful to define $p = \bar{p} + e_p$, $w = \bar{w} + e_w$, with $E(p) = \bar{p}$ and $E(w) = \bar{w}$. Thus, we also have $R = py = \bar{p}y + e_p y \equiv \bar{R} + e_R$, with $E(R) = \bar{R}$.

We assume that the firm maximizes expected utility of profits, $U(\pi) = U(R - wx)$, where U is a continuous and increasing Von Neumann-Morgenstern utility function. The firm's problem is given by:

$$Max_{y,x}\ \{\int_{(R,w)\in\mathcal{A}} U[R - wx]dG : (x, y) \in \mathcal{T}\} \tag{1}$$

This problem can be solved in two steps. First, we define the restricted indirect utility functional, J, as the solution to the problem:

$$Max_x\ \{\int_{(R,w)\in\mathcal{A}} U[R - wx]dG : (x, y) \in \mathcal{T}\} \equiv J(G, y) \tag{2}$$

Second, we solve for the optimal level of output,

$$Max_y\ J(G, y) \tag{3}$$

The solution to the two step problem is the same as the solution to problem (1).

In order to avoid having to make specific assumptions regarding output market structure in the empirical analysis[3], we will focus on the first step of the problem, as in (2). This corresponds to the standard cost minimization problem in the theory of the firm without uncertainty and will facilitate comparison between the two models. To simplify the notation, we write $J(G)$, instead of $J(G, y)$, whenever y is not discussed explicitly.

Given that $J(G)$ can be written as $J(G) \equiv Max_x\ \ \{J(G, x) : (x, y) \in \mathcal{T}\}$, where (the conditional preference functional) $J(G, x) \equiv E[U[R - wx]]$, it has similar properties to the usual support (dual) functions[4] and can be used to obtain duality results. Specifically, since (i) the constraint set, $\mathcal{T}$, is non-empty and compact, (ii) U is continuous and increasing in profits[5] and (iii) the conditional functional $J(G, x)$ has a compact range and is continuous in x, it follows from the Theorem of the Maximum (Berg (1963))[6], that J is continuous and the optimal solution for x is upper semi-continuous. Given the "linearity in probabilities" of expected utility, J is also convex in G.[7] Machina (1984) (p. 208, Theorems 2 and 3.) also shows that convexity and continuity completely characterize the functional J, so that any functional

[3]For empirical applications of models with non-competitive markets see Appelbaum (1979), (1982).
[4]This is pointed out in Machina (1984). For a discussion of support functions, see Rockafellar (1970).
[5]Hence continuous and increasing /decreasiong in (R, w), respectively.
[6]This is also given in Theorem 1 in Machina (1984) , p. 206.
[7]Proof can be supplied upon request.

with these properties is the indirect utility functional for some preferences. Underlying utility functions are, therefore, "recoverable" and moreover, the "recovered preferences" are the same as the actual ones.

2.2 Moments and the Indirect Utility Function

While the indirect functional $J(G)$ is useful in obtaining duality relationships, it is not the only possible characterization of indirect utility, nor is it the most convenient one to use empirically. Instead of looking at $J(G)$, it is possible to consider other characterizations of indirect preferences. For example, since the random variables (R, w) are bounded, the moments of G exist and uniquely determine the distribution G[8]. Thus, if we denote the moments by m, the indirect utility can be written in terms of the moments as:

$$J(G) = J(\hat{G}(m)) \equiv V(m) \tag{4}$$

Given that J is continuous and convex in G, it follows that V is continuous and convex in the moments.[9]

Since the purpose of this paper is to provide a framework for empirical applications, it is both convenient and natural to work with an indirect utility function defined over moments. First of all, it is very simple to obtain the firm's input demand functions directly, by applying the envelop theorem to a function of the moments. Second, moments are much easier to calculate empirically. Thus, given that the moments uniquely characterize the distribution G, the solution to problem (2) can be described by an indirect utility function, V, that is defined over these moments:

$$Max_x\{\int_{(R,w)\in\mathcal{A}} U[R - wx]dG : (x, y) \in \mathcal{T}\} \equiv J(G, y) = V(\bar{w}, \bar{R}, y, \Sigma, \delta) \tag{5}$$

where Σ is the covariance matrix of the random variables (R, e_w) and the vector δ represents moments of higher order.

The firm's input demand functions can be easily derived from the indirect utility function, V. First, since $w = \bar{w} + e_w$, and $R = \bar{R} + e_R$we get from (5):

$$\frac{\partial V}{\partial \bar{w}_i} = -E[U'(\pi)]x_i \tag{6}$$

$$\frac{\partial V}{\partial \bar{R}} = E[U'(\pi)]^{10} \tag{7}$$

[8]Bounded support is a sufficient condition for the distribution function to be uniquely determined the moments (this is the so called "moments problem"). See, for example, Wilks (1964), theorem 5.5.1. p. 126.

[9]Proof can be supplied upon request.

[10]Note that the derivative in (7) is to be understood as the effect of an exogenous change in the mean of R (of some shift parameter), which is not due to a change in output.

The input demand functions are, therefore, given by (the equivalent of) Roy's identity as

$$x_i = -\frac{\partial V}{\partial \bar{w}_i} / \frac{\partial V}{\partial \bar{R}}. \tag{8}$$

Given a functional form for V these functions can be easily derived.

Without uncertainty, additional properties of dual functions can be easily obtained, since either the objective function (in the theory of the firm), or the constraint (in the theory of the consumer) are linear. Specifically, dual functions usually also satisfy homogeneity and monotonicity restrictions[11]. These restrictions are used to obtain qualitative results, to test the underlying theory and to reduce the number of parameters that need to be estimated in empirical applications.

Since here profits are transformed nonlinearly by the utility function, similar properties do not necessarily hold for the indirect preference function V. In general, the properties of V depend on the properties of the utility, production and the density functions. It is still possible, however, to derive additional properties that V and the corresponding input demand functions must satisfy.

1. <u>The Slutsky Equation</u>**:**

Following Chavas and Pope (1985) and Dalal (1990) (where output price uncertainty is examined), define the compensated substitution effect (holding maximum expected utility, V, constant) of input i with respect to expected input price j, S_i^j, as

$$S_i^j \equiv \frac{\partial x_i}{\partial \bar{w}_j} |_{dV=0} = \frac{\partial x_i}{\partial \bar{w}_j} + \frac{\partial x_i}{\partial \bar{R}} \frac{\partial \bar{R}}{\partial \bar{w}_j} |_{dV=0} \tag{9}$$

By substituting $\frac{\partial \bar{R}}{\partial \bar{w}_j} |_{dV=0} = x_j$, into (9), we get the equivalent of the Slutsky equation in consumer theory as:

$$\frac{\partial x_i}{\partial \bar{w}_j} = \frac{\partial x_i}{\partial \bar{w}_j} |_{dV=0} - x_j \frac{\partial x_i}{\partial \bar{R}} \tag{10}$$

Since $S_i^j = -[V_{ij} + V_{\bar{R}j} x_i + V_{\bar{R}i} x_j + x_i x_j V_{\bar{R}\bar{R}}]/V_{\bar{R}}$, the $[S_i^j]$ matrix is symmetric. In addition, it can be shown to be negative semi-definite[12]. The convexity of V in moments and the fact that the matrix $[S_i^j]$ is (symmetric) negative semi-definite provide us with a set of curvature restrictions that must be satisfied by V.

It is worth noting, however, that the matrix $\frac{\partial x_i}{\partial \bar{w}_j}$ itself is not necessarily negative semi-definite. Thus, input demand functions are not necessarily downward sloping.[13] This corre-

[11] See Diewert (1982), Epstein (1981), for a discussion of these properties.

[12] The proof is similar to the one given in Chavas (1985) and will be supplied upon request.

[13] For this to be the case it is necessary to impose additional restrictions on the underlying production and utility functions. For example, with known input prices, but an uncertain output price, it was shown (See Sandmo (1971), Batra and Ullah (1974), Hartman (1976)), that with decreasing absolute risk aversion, $\partial x_i / \partial R > 0$, so that the "income effect" has the same sign as S_i^i.

sponds to the standard result in consumer theory, where Marshallian demand functions are not necessarily downward sloping, but compensated demands always are.

2. Monotonicity:

(i) Since $E[U'(\pi)] > 0$, we get from (6) and (7) that V is non-increasing in expected input prices, but non-decreasing in expected revenue.

(ii) From (5) we get

$$\frac{\partial V}{\partial \sigma_i} = -E[U'(\pi)\epsilon_i]x_i = -x_i Cov[U'(\pi), \epsilon_i] \tag{11}$$

$$\frac{\partial V}{\partial \sigma_R} = E[U'(\pi)\epsilon_R] = Cov[U'(\pi), \epsilon_R] \tag{12}$$

where $\sigma_i \equiv \sqrt{var(w_i)}$, $\sigma_R \equiv \sqrt{var(R)}$ and ϵ_i, ϵ_R are the normalized random variables, defined by: $\epsilon_i \equiv e_i/\sigma_i$, $\epsilon_R \equiv e_R/\sigma_R$. Since the random variables ϵ_R, ϵ_i may be either positively, or negatively correlated, the sign of U'' is not enough to determine the signs of the covariance terms in (11) and (12) [14].

(ii). It is well known that with risk neutrality (or without uncertainty), changes in $\bar{R}$, σ_w and σ_R, do not affect demand. Conversely, if such changes do affect demand, the firm cannot be risk neutral[15]. These properties provide a simple econometric test for risk neutrality.

3. Homogeneity:

Since $U(\lambda\pi) \neq \lambda U(\pi)$, unless U is linear (and since $\pi = \bar{R} + \sigma_R\epsilon_R - \bar{w}x - \sigma_w\epsilon_w$ is linear in $\bar{R}, \bar{w}, \sigma_R, \sigma_R$), V cannot be homogeneous of degree one in $(\bar{w}, \bar{R}, \sigma_R, \sigma_w)$, or in $(\bar{w}, \bar{R})$. A necessary and sufficient condition for linear homogeneity of V in $(\bar{w}, \bar{R})$ is that we have risk neutrality[16].

What can be said about the homogeneity properties of input demand functions? If V is linear homogeneous (with risk neutrality), the demand functions are homogeneous of degree zero (in expected prices). On the other hand, when V is not linear homogeneous (without risk neutrality), input demand functions are not homogeneous of degree zero in expected prices[17].

[14] To see this, assume that there are only two random variables, say w_1 and R, which are distributed according to a bivariate normal distribution. Then, applying Stein's Lemma we can write the two covariance terms in (11) and (12) as: $Cov(U'(\pi), \epsilon_1) = \theta_1 E\{U''(\pi)\}$ and $Cov(U'(\pi), \epsilon_R) = \theta_R E\{U''(\pi)\}$, where, $\theta_1 \equiv [\sigma_R Cov(\epsilon_1, \epsilon_R) - x_1\sigma_1]$ and $\theta_R \equiv [\sigma_R - x_1\sigma_1 Cov(\epsilon_1, \epsilon_R)]$. Since we do not know the signs of θ_1 and θ_R, we do not know if V is increasing, or decreasing in σ_1 and σ_R, even if we have risk aversion. This problem does not arise with one random variable, or when $Cov(\epsilon_1, \epsilon_R)$.

[15] It should be noted that with constant absolute risk aversion (CARA) we also have $\frac{\partial x_i}{\partial \bar{R}} = 0$. Thus, $\frac{\partial x_i}{\partial \bar{R}} \neq 0$ also implies the rejection of CARA.

[16] With risk neutrality, $E(U) = \bar{R} - \bar{w}x$, which is linear in $(\bar{w}, \bar{R})$, thus leading to linear homogeneity of V.

[17] To see this, note that the first order conditions for problem (5) yield that $\frac{\beta_i + \bar{w}_i}{\beta_j + \bar{w}_j} = \frac{F_i}{F_j}$, for the required level of output, where $\beta_h = \frac{Cov(U', w_h)}{E(U')}$, $h = i, j$. For a general utility function, the input mix and hence (with a given level of output) also input levels will be unaffected by proportionate changes in expected prices if and only if $\beta_i = \beta_j = 0$, for all i, j, which is the case under risk neutrality. Thus, in general, input demands are not zero homogeneous in expected prices. Note that this means that (since $V_i/V_j = x_i/x_j$) the slopes of the level surfaces of V along a ray through the origin are not constant, in other words, V, cannot be homothetic either.

4. symmetry:
From (8) we get

$$\begin{aligned} \frac{\partial x_i}{\partial \bar{w}_j} &= -\{V_{ij}V_{\bar{R}} - V_{\bar{R}j}V_i\}/V_{\bar{R}}^2 \\ \frac{\partial x_j}{\partial \bar{w}_i} &= -\{V_{ij}V_{\bar{R}} - V_{\bar{R}i}V_j\}/V_{\bar{R}}^2 \end{aligned} \tag{13}$$

Thus, input demand functions are symmetric ($\frac{\partial x_i}{\partial \bar{w}_j} = \frac{\partial x_j}{\partial \bar{w}_i}$) if and only if

$$V_{\bar{R}j}V_i = V_{\bar{R}i}V_j \tag{14}$$

This condition can be written as $\partial \ln(V_i/V_j)\partial\bar{R} = 0$, or $\partial(V_i/V_j)\partial\bar{R} = 0$, which is the condition that w_i, w_j are weakly separable from $\bar{R}$[18]. Equivalently, since $V_i/V_j = x_i/x_j$, this condition is the requirement that input proportions are unaffected by expected revenues.

3 Empirical Implementation

3.1 Econometric Specification

Having discussed the theoretical framework, we now provide an example of an empirical application of our model. To implement the model empirically, we first have to specify a functional form for the indirect utility function $V(m)$. Given this functional form, if the moments of $G(R, w)$ (or $G'(p, w)$) are known, then we could simply estimate the system of equations (8). Unfortunately, the moments of the distribution are generally not known and will, therefore, have to be estimated. For example, assuming rational expectations, the firm forms its expectations of the moments of G' by estimating them from market information. Given estimates of $\bar{p}$, $\bar{w}$, and Σ (and possibly higher moments), we can estimate the firm's input demand functions. The model can then be used to test hypotheses regarding attitudes toward risk and to estimate the effects of uncertainty.

The example we provide applies the model to the U.S. manufacturing industry, using the Berndt and Wood (1986) data, for the period 1947-1981[19]. We choose this data set for two reasons. First, it is a well known data set that has been extensively used in the literature. Second, it provides a possible framework for the analysis of the effects of energy price shocks[20].

[18]See Blackorby, Primont and Russel, (1979), pages 52 and 67.
[19]See Berndt and Wood (1975), (1986) for a discussion of the data construction.
[20]A drawback with this application is that, like many applications in applied production theory, it applies a theoretical model which has been developed at the firm level, to aggregate data, thus assuming away the problems of aggregation. Hopefully, with better data, an application at the micro level will be possible in the future.

This application, however, should only be viewed as an example of the method proposed in the paper.

To reduce the dimensions of the model we aggregate energy and materials into a single input. Thus, we assume that there are three competitively priced inputs in the production process: capital, x_k, labour, x_l and intermediate inputs (energy and materials) x_m, whose prices are w_k, w_l and w_m respectively. We use a Divisia index to construct the aggregate price and quantity of the intermediate input. We assume that the prices of the output and intermediate inputs are uncertain when decisions are made, whereas other prices are known.

We assume that the inverse indirect utility function, $I = 1/V$, is given by the translog function[21]

$$\begin{aligned} \ln I = -\ln V &= a_0 + \sum_i a_i \ln i + \frac{1}{2}\sum_i \sum_j a_{ij} \ln i \ln j \\ & i, j = w_k, w_l, \bar{w}_m, \bar{R}, y, \sigma_R^2, \sigma_m^2, \sigma_{Rm} \end{aligned} \tag{15}$$

, with the symmetry restrictions: $a_{ij} = a_{ji}$. Applying Roy's identity, we get the input "revenue share" equations as:[22]

$$\begin{aligned} -s_i &= \frac{a_i + \sum_j a_{ij} \ln j}{a_r + \sum_j a_{rj} \ln j} \\ j &= w_k, w_l, \bar{w}_m, \bar{R}, y, \sigma_R^2, \sigma_m^2, \sigma_{Rm} \\ i &= w_k, w_l, \bar{w}_m \end{aligned} \tag{16}$$

where $s_i \equiv w_i x_i / \bar{R}$ for $i = k, l$ and $s_m \equiv \bar{w}_m x_m / \bar{R}$. Since the input demand equations (16) are homogeneous of degree zero in the parameters we use the normalization

$$a_r = -1 \tag{17}$$

For empirical implementation the model has to be imbedded within a stochastic framework. To do this we assume that equations (16) are stochastic due to "errors in optimization". We define the "optimization errors" in the demand equations at time t as $v_k(t), v_l(t)$ and $v_m(t)$. We denote the column vector of disturbances at time t as $v(t) \equiv \{v_k(t),\ v_l(t),\ v_m(t)\}$ and assume that the vector of disturbances is identically and independently, joint normally distributed

[21] Since the empirical application assumes normality, we do not include higher order moments in (18). For other distributions it is possible to calculate higher moments and include them in the function V. For examples of models with skewness, see Roll and Ross (1983) and Singleton and Wingender (1986).

[22] Since $\ln I = -\ln V$, we have $x_i = -I_i/I_{\bar{R}} = -V_i/V_{\bar{R}}$. Thus, $\frac{\partial \ln I}{\partial \ln \bar{w}_i} / \frac{\partial \ln I}{\partial \ln \bar{R}} = \frac{I_i \bar{w}_i}{I_{\bar{R}} \bar{R}} = -\frac{x_i \bar{w}_i}{\bar{R}} = -s_i$. Also note that for the four inputs, we use interchangeably: $\alpha_k = \alpha_{w_k}$,$\alpha_m = \alpha_{w_m}$,$\alpha_l = \alpha_{w_l}$.

with mean zero and non-singular covariance matrix Ω

$$E[v(s)v(t)] = \begin{cases} \Omega \ \forall \ s,t \ \ if \ \ s = t \\ 0 \quad if \quad t \neq s \end{cases} \tag{18}$$

where Ω is a 3×3 positive definite matrix.

3.2 Empirical Results

First, we estimate the means, variances and correlation of w_m and p. We specify the prices p and w_m as given by the equations

$$w_m(t) = \gamma_m + \gamma_{mm} w_m(t-1) + \gamma_{mp} p(t-1) + \gamma_{mt} t + e_m(t) \tag{19}$$

$$p(t) = \gamma_p + \gamma_{mp} w_m(t-1) + \gamma_{pp} p(t-1) + \gamma_{pt} t + e_p(t) \tag{20}$$

where e_p, e_m are distributed according to a bivariate normal distribution[23] with $E(e_p) = E(e_m) = 0$ and a covariance matrix S, with: $Var(e_p) = \sigma_p^2, Var(e_m) = \sigma_m^2$ and $Cov(e_p, e_m) = \sigma_{pm}$. To obtain time varying values for the covariance matrix, the errors are assumed to follow the multivariate ARCH(1) process given by:

$$\begin{aligned} \sigma_p^2(t) &= \beta_p + \beta_p^1 e_p^2(t-1) \\ \sigma_m^2(t) &= \beta_m + \beta_m^1 e_m^2(t-1) \\ \sigma_{pm}(t) &= \beta_{pm} + \beta_{pm}^1 e_p(t-1) e_m(t-1) \end{aligned} \tag{21}$$

A multivariate ARCH (MARCH) specification similar to the one in (21) has been introduced by Engle, Granger and Kraft (1984) and has been applied in several studies of financial markets[24]. In addition to the variances following an ARCH process, the specification in (21) also allows the covariance between the prices of m and p to be autoregressive. The MARCH process has been useful in finance applications, where portfolio diversification and the correlation between the returns on assets play an important role. It is equally useful in modelling firm behaviour, when there is more than one source of uncertainty.

We use the maximum likelihood technique to estimate the two price equation (19) and (20), subject to the MARCH(1) specification in (21). Using the parameter estimates for equations (19) and (20), we obtain the estimates of the means and covariance matrix of the expected prices of intermediate inputs and output. We use these to calculate the estimated mean of R and the covariance matrix of w_m and R; Σ. The parameter estimates for equations (19), (20)

[23]Note that the normal distribution is uniquely determine by the means and covariance matrix. Hence, in this case, a finite number of moments completely characterizes the distribution function G.

[24]See Bollerslev, Engle and Wooldridge (1988), Baillie and Myers (1991). See also Bera and Higgins (1993) for a survey of multivariate ARCH.

and the corresponding estimates of the first two moments of intermediate inputs and output prices and their correlation are given in Table 1 and Table 2 respectively[25].

Given the estimates of $\bar{w}_m$, $\bar{R}$, σ_m^2, σ_R^2 and ρ, we estimate equations (16), using the iterative Zellner technique, with symmetry ($\alpha_{ij}= \alpha_{ji}$) and the normalization ($\alpha_r = -1$) restrictions imposed[26]. The parameter estimates are given in Table 3. Using the parameter estimates, we first verify that the regularity conditions are satisfied at the point of approximation. We find that the conditions for monotonicity and convexity of I (or V) and the semi-definiteness of the compensated substitution matrix (S_{ij}) are satisfied at the point of approximation[27]. We also found that the fitted values of the revenue shares of all inputs are between zero and one and their sum is less then 1.

We test for linear homogeneity in of V in $\bar{w}, \bar{R}$ and reject it. thus, risk neutrality must be rejected. Next, we test for the necessary and sufficient conditions for symmetry (the weak separability restrictions). From (15) we get that local (at the point of approximation) weak separability of input prices i, j from $\bar{R}$ (condition (14) is satisfied if and only if[28]

$$\alpha_i \alpha_{jR} = \alpha_{iR} \alpha_j \tag{22}$$

We test for these three local weak separability restrictions, using the likelihood ratio test. We get that $\chi^2 = 10.67$, thus, since $\chi^2_{(3,.01)} = 11.3$, we cannot reject the hypothesis that all inputs are locally symmetric (at the 1%, significance level)[29].

To examine the effects of output price (or revenue) on input demand we consider the output price elasticities, which are given by

$$\mu_{i\bar{p}} \equiv \mu_{iR} = \alpha_{iR}/\delta_i - \alpha_{RR}/\delta_R + 1 \quad for \quad i = k, l, m \tag{23}$$

where

$$\delta_i \equiv -\frac{\partial \ln I}{\partial \ln i} \equiv \frac{\partial \ln J}{\partial \ln i} = a_i + \sum_j a_{ij} \ln j \qquad i = w_k, w_l, \bar{w}_m, \bar{R}$$
$$j = w_k, w_l, \bar{w}_m, \bar{R}, y, \sigma_R^2, \sigma_m^2, \sigma_{Rm} \tag{24}$$

[25] As Table 1 indicates, the MARCH(1) hypothesis could not be rejected.

[26] To further reduce the number of parameters, we also assume in the estimation that the restrictions: $\alpha_{\bar{R}\rho} = \alpha_{\bar{R}\sigma_m} = \alpha_{\bar{R}\sigma_{\bar{R}}} = 0$, hold. These restrictions were tested for and could not be rejected.

[27] we also estimated the model using the instrumental variables procedure suggested by Pagan (1984) and Pagan and Ullah (1988). Using the estimates of $\bar{w}_m$, $\bar{R}$, σ_m^2, σ_R^2 and σ_{Rm}^2, we estimated equations (16), by the instrumental variables three stage least squares procedure, using the instrumental variables from Kohli (1991). As is shown in Pagan and Ullah (1988), this instrumental variable method leads to consistent estimates and reliable standard errors for inference purposes. The results of this model were similar as far as the regularity conditions are concerned.

[28] Weak separability requires that $(\alpha_i \alpha_{jR} - \alpha_j \alpha_{iR}) + \sum_n [\alpha_{in} \alpha_{jR} - \alpha_{jn} \alpha_{iR}] \ln n = 0$, for all $n = w_k, w_l, \bar{w}_m, \bar{R}, \sigma_m^2, \sigma_R^2, \sigma_{Rm}, y$ and $i, j = w_k, w_l, \bar{w}_m$. Global weak separability (at all data points), therefore, requires that $\alpha_i/\alpha_j = \alpha_{iR}/\alpha_{jR} = \alpha_{in}/\alpha_{jn}$ for all $n = w_k, w_l, \bar{w}_m, \bar{R}, \sigma_m^2, \sigma_R^2, \sigma_{Rm}, y$, whereas locally, the requirement is that: $\alpha_i \alpha_{jR} - \alpha_j \alpha_{iR} = 0$. See Blackorby, Primont and Russel, (1979), pp. 297-300, for a discussion of these conditions.

[29] But, we reject global symmetry of all input demand functions.

Input demands are unaffected by $\bar{p}$ (or $\bar{R}$), if and only if $\mu_{i\bar{p}} = \mu_{i\bar{R}} = 0$ for all $i = w_k, w_l, \bar{w}_m$. Thus, for local neutrality of input demand with respect to $\bar{R}$, we require that

$$\alpha_{iR} + \alpha_i \alpha_{RR} + \alpha_i = 0, \quad i = w_k, w_l, \bar{w}_m \tag{25}$$

We tested for these restrictions and rejected the hypothesis that, locally, revenue does not affect all input demands ($\chi^2 = 11.94, \ \ \chi^2_{(3,.01)} = 11.3$)[30]. Thus, again, we conclude that risk neutrality must be rejected[31].

Having rejected risk neutrality, can we go further and determine that we actually have risk aversion? One way to check if there is risk aversion is to see whether the indirect utility function is decreasing in variances. Unfortunately, all parameters which do not involve $w_k, w_l, \bar{w}_m$ or $\bar{R}$, are lost in the differentiation of the indirect utility function. Consequently, some of the parameters involving the variances do not appear in our estimated system, which means that we cannot check for monotonicity in the second moments. It is, however, possible to check for risk aversion indirectly. To see this, note that since only w_m and R are random, we can write the firms problem as

$$Max_{x_m} E[U(R - w_m x_m - C(w_k, w_l, x_m, y)] \tag{26}$$

where $C(w_k, w_l, x_m, y) \equiv Min_{x_k, x_l}(w_k x_k + w_l x_l : (x_k, x_l, x_m, y) \in \mathcal{T})$ is a standard restricted cost function. The solution to problem (26) can be written as:

$$\bar{w}_m + \Delta = \partial C / \partial x_m \tag{27}$$

where $\Delta \equiv Cov(U'(\pi), \epsilon_m)/E(U') = [\sigma_R Cov(\epsilon_m, \epsilon_R) - x_m \sigma_m] E[U''(\pi)]/E[U'(\pi)]$ (as in footnote (13)). Since under certainty (or risk neutrality), $\Delta = 0$, x_m will be lower under uncertainty compared with the certainty (or risk neutrality) case if and only if $\Delta > 0$. It is, therefore, possible to sign Δ, by comparing the levels of x_m with and without uncertainty (or with risk neutrality). To obtain the values for x_m under certainty we note that without uncertainty, the indirect utility function becomes the usual profit function, which is linear homogeneous in $w, \bar{R}$. This is obtained by imposing linear homogeneity (in $w, \bar{R}$) and the requirement that $\partial I / \partial \bar{R} = -1, \ \alpha_{ij} = 0$ for all $j = \sigma_m, \sigma_R, \sigma_{Rm}$. In comparing the cases under certainty and uncertainty we follow the standard practice and take the certainty prices as their expected values. Thus, we estimate the model with these restrictions imposed and use the parameter estimates to calculate the predicted values of x_m. Comparing the estimated values of x_m with and without uncertainty, we find that x_m in the restricted case (no uncertainty) is higher than under the unrestricted case (with uncertainty) for all observations. Thus, we conclude that we

[30] Since global restrictions are stronger, this implies that global neutrality will also be rejected.

[31] Note, however, that the converse is not necessarily true. In other words, if these restrictions cannot be rejected, it does not follow that we have risk neutrality. For example, we may have CARA.

must have $\Delta > 0$. What does this implies as far as risk aversion is concerned? To see this, we use the estimated covariance matrix to calculate term, $\theta_m \equiv [\sigma_R Cov(\epsilon_m, \epsilon_R) - x_m \sigma_m]$ and find it to be negative at all sample points. Thus, since $\Delta > 0$, we must have $E[U''(\pi)] < 0$. Hence (If U'' does not change sign) we must have risk aversion. The fact that $\Delta > 0$ implies that the indirect utility function is decreasing in σ_m. We also calculate $\theta_R \equiv [\sigma_R - x_m \sigma_m Cov(\epsilon_m, \epsilon_R)]$ and find it to be negative at all sample points. The indirect utility function is, therefore, increasing in σ_R.

Can we go even further and say something about the type of risk aversion present? From (7) we can get:

$$\frac{\partial^2 V}{\partial \overline{R}^2} = E(U'') + \frac{\partial x_m}{\partial \overline{R}} E[U'' \epsilon_m) - \theta_m \frac{E(U'')}{E(U')} \tag{28}$$

We found above that V is locally convex in $\overline{R}$. Calculating the output price elasticity of x_m, we find that it (and hence also $\frac{\partial x_m}{\partial \overline{R}}$) is globally (significantly) positive. Thus, since, $\theta_m < 0$, $E(U'') < 0$, $\frac{\partial x_m}{\partial \overline{R}} > 0$, $E(U') > 0$, we must have $E[U'' \epsilon_m) > 0$, which is consistent with decreasing absolute risk aversion.

The input price elasticities are given by:

$$\mu_{ij} = \alpha_{ij}/\delta_i - \alpha_{Rj}/\delta_R \quad for \quad i \neq j \tag{29}$$

$$\mu_{ii} = \alpha_{ii}/\delta_i - \alpha_{Ri}/\delta_R - 1 \tag{30}$$

Given the parameter estimates we calculate the local input demand elasticities and report them in Table 4. As Table 4 shows, the own price elasticities are, locally, negative for all inputs. All input demand functions are, therefore, locally negatively sloped.

Cross price elasticities of demand are usually not symmetric. Under uncertainty, however, the cross price elasticities of demand, μ_{ij} and μ_{ji}, do not even have to have the same sign[32]. Indeed, as Table 4 shows, not all the demand elasticities are "sign symmetric". Specifically, capital and intermediate inputs are mutually complements (as was found by Berndt and Wood (1975) for capital and energy), and the same is true for intermediate inputs and labour. Capital and labour, however, are not "sign symmetric".

To examine the effects of price variances on input demand, we get the "uncertainty elasticities" as

$$\mu_{i\sigma_n} = \alpha_{i\sigma_n}/\delta_i - \alpha_{R\sigma_n}/\delta_R \quad for \quad i = k, l, m, \quad n = m, R \tag{31}$$

[32]To see this, assume that $\mu_{ij} > 0$. Then, we must have $\alpha_{ij} > \frac{\alpha_{Rj}\delta_i}{\delta_R} = [\frac{\alpha_{Ri}\delta_j}{\delta_R}][\frac{\alpha_{Rj}\delta_i}{\alpha_{Ri}\delta_j}]$. But, given this, it is still possible to have $\alpha_{ij} < \frac{\alpha_{Ri}\delta_j}{\delta_R}$, which implies that $\mu_{ji} < 0$.

Input demands are locally unaffected by σ_n^2, $n = m, R$, if and only if $\mu_{i\sigma_n} = 0$ for all $i = k, l, m$ and $n = m, R.$. These (local) restrictions can be written as

$$\alpha_i \alpha_{R\sigma_n} = -\alpha_{i\sigma_n} \quad n = m, R \;\; and \quad i = k, l, m \tag{32}$$

First, we test and reject the hypothesis that the (six) restrictions in (32), hold for all inputs (i.e., for all $i = k, l, m$ and $n = m, R$). We get $\chi^2 = 17.3$ and since $\chi^2_{(6,.01)} = 16.8$, we reject the hypothesis that all inputs are locally unaffected by uncertainty[33]. Given that we reject that uncertainty does not affect input demands, again, risk neutrality must be rejected.

The estimated uncertainty elasticities, calculated at the point of approximation, and their standard errors are reported in Table 4. As we can see, the effects of price uncertainty on input demand is quite small.

Finally, we calculate the covariance elasticities and report them also in Table 4. As these figures indicate an increase in the covariance between output and intermediate inputs prices will have a significant positive effect on the demand for labour and capital, but an insignificant effect on the demand for intermediate inputs.

4 Conclusion

This paper provides an empirically implementable framework for the analysis of firms' behaviour under uncertainty. Applying the indirect (expected) utility function, we develop a model which can be used to capture the effects of (input and output) price uncertainty on firms' behaviour. The model is used to test implications of the theory, to calculate the effects uncertainty on input demand and to calculate price elasticities.

We provide an example in which we estimate the effects of output and intermediate inputs (energy) price uncertainty on input demand by the U.S. manufacturing industry. We test for and reject risk neutrality. Specifically, we find that production responses indicate the existence of risk aversion and are consistent with behaviour under decreasing absolute risk aversion. The actual effects of uncertain prices on input demand were, however, rather small.

[33]Given that the local restrictions are rejected it is clear that the global restrictions will be rejected as well.

5 References

Antonovitz, F. and T. Roe, (1986), "A Theoretical and Empirical Approach to the Value of Information in Risky Markets," Review of Economics and Statistics, 68, 105-114.

Appelbaum, E., (1979), "Testing Price Taking Behaviour," Journal of Econometrics.

Appelbaum, E., (1982), "The Estimation of the Degree of Oligopoly Power," Journal of Econometrics, 287-299.

Appelbaum, E. and E. Katz, (1986), "Measures of Risk Aversion and the Comparative Statics of Industry Equilibrium," American Economic Review, 76.

Appelbaum, E., (1991), "Uncertainty and the Measurement of Productivity," Journal of Productivity Analysis, 2, 157-170.

Appelbaum, E. and U. Kohli, (1995), "Import Price Uncertainty and the Distribution of Income", Review of Economics and Statistics, forthcoming.

Baillie, R.T. and R.J. Myers, (1991), "Bivariate GARCH estimation of Optimal Commodity Futures Hedge", Journal of Applied Econometrics, 16,109-124.

Batra, R. and A. Ullah, (1974), "Competitive Firm and the Theory of Input Demand Under Price Uncertainty," Journal of Political Economy, 82, 537-548.

Bera, A.K. and M.L. Higgins, (1993), "Arch Models: Properties, Estimation and Testing", Journal of Economic Surveys, 7, 307-366.

Berg, C., (1963), Topological Spaces, tr. by Patterson, New York, MacMillan.

Berndt E.R. and D.O. Wood, (1975), "Technology, Prices and Derived Demand for Energy," Review of Economics and Statistics, 57, 376-84.

Berndt E.R. and D.O. Wood, (1986), "U.S. Manufacturing Output and Factor Input Price and Quantity Series, 1908-1947 and 1947-1981," Energy Laboratory working paper, 1986-010 wp, MIT.

Blackorby, C., Primont, D. and R. Russel, (1979), Duality, Separability and Functional Structure: Theory and Economic applications, Amsterdam: North-Holland.

Bollerslev, T., R.F. Engle and J.M. Wooldridge, (1988), "A Capital Asset Pricing Model with Time-Varying Covariances", Journal of Political Economy, 96, 116-131.

Chavas, J-P, (1985), "On the Theory of the Competitive Firm under Uncertainty when Initial Wealth is Random," Southern Economic Journal, 51, 818-827. 223-235.

Chavas, J-P, and R. Pope, (1985), "Price Uncertainty and Competitive Firm Behavior: Testable Hypotheses from Expected Utility Maximization," Journal of Economics and Business, 37, 223-235.

Dalal, A. J., (1990), "Symmetry Restrictions in the Analysis of the Competitive Firm Under Price Uncertainty," International Economic Review.

Diewert, W.E., (1982), "Duality Approaches in Microeconomics," in K.J. Arrow and M.D.

Intrilligator, eds. Handbook of Mathematical Economics, Vol. 2, (North-Holland, Amsterdam).

Engle, R.F., C.W. Granger and D. Kraft, (1986), "Combining Competing Forecasts of Inflation Using a Bivariate ARCH Model", Journal of Economic Dynamics and Control, 8, 151-165.

Epstein, L., (1978), "Production Flexibility and the Behaviour of the Competitive Firm Under Price Uncertainty," Review of Economic Studies, 45, 251-261.

Epstein, L., (1980) "Multivariate Risk Independence and Functional Forms for Preferences and Technologies", Econometrica, 84, 973-985.

Epstein, L., (1981) "Generalized Duality and Integrability", Econometrica, 49, 655-678.

Feller, William, (1966), An Introduction to Probability Theory and Its Applications, John Wiley & Sons, New York.

Fuss, M. and D. McFadden, (eds.), (1978), Production Economics: A Dual Approach to Theory and Applications, Amsterdam: North-Holland.

Hartman, R., (1976), "Factor Demand with Output Price Uncertainty", American Economic Review, 66, 675-681.

Jorgenson, D.W., (1986), "Econometrics Methods for Modelling Producer Behavior", in Z. Griliches and M.D. Intriligator (eds.), Handbook of Econometrics, Vol. 3, Amsterdam, North-Holland, 1841-1915.

Jorgenson, D.W., F.M. Gollop and B. Fraumeni, (1987), Productivity and U.S. Economic Growth, North Holland.

Just, R., (1974), "An Investigation of the Importance of Risk in Farmers' Decisions," American Journal of Agricultural Economics, 56, 14-25.

Machina, M., (1984), "Temporal Risk and Induced Preferences", Journal of Economic Theory, 33, pp. 199-231.

Pagan, A., (1984), "Econometric Issues in the Analysis of Regressions with Generated Regressors", International Economic Review, 25, 1.

Pagan, A. and A. Ullah, (1988), "Econometric Analysis of Models with Risk Terms" Journal of Applied Econometrics.

Parkin, M., (1970), "Discount House Portfolio and Debt Selection," Review of Economic Studies, 37, 469-497.

Pope, R.D., (1980), "The generalized Envelope Theorem and Price Uncertainty," Southern Economic Journal, 21, 75-86.

Rao, C.R., (1973), Linear Statistical Inference and Its Applications, John Wiley & Sons, New York.

Rockafellar, R.T., (1970), Convex Analysis, Princeton University Press, Princeton, N.J.

Roll, R. and S.A. Ross, (1983), "An Empirical Investigation of Arbitrage Pricing Theory," Journal of Finance, 35, 1073-1104.

Sandmo, A., (1971), "On the Theory of the Competitive Firm Under Price Uncertainty," American Economic Review, 61 65-73.

Silberberg, E., (1978), The Structure of Economics: A Mathematical Analysis (New York: McGraw-Hill, 1978).

Singleton, J.C. and J. Wingender, (1986), "Skewness Persistence in Common Stock Returns" Journal of Financial and Quantitative Analysis, 335-341.

Wilks, S.S, (1964), Mathematical Statistics, John Wiley & Sons, New York.

Table 1: Parameter Estimates of Price Equations

Parameter	Estimate	t-Statistics
γ_m	.159424	1.35073
γ_{mp}	-.894922	-1.40833
γ_{mm}	.73270	3.53694
γ_{mt}	-.024098	-.555652
γ_p	.244183	3.38376
γ_{pp}	-.532625	-1.26564
γ_{pm}	1.25144	3.86655
γ_{pt}	.246517E-02	.070244
β_{pm}	.882522	7.04975
β_{pm}^1	.113125	2.665867
β_p	.00021	2.56893
β_p^1	.959959	3.95019
β_m	.00029	1.79243
β_m^1	1.182808	3.97396

Table 2 Moments of Intermediate Input and Output Prices

year	$\bar{w}_m$	σ_m	$\bar{p}$	σ_p	$\rho \equiv \frac{\sigma_{rm}}{\sigma_p \sigma_m}$
49	0.60758	0.025736	0.64329	0.032919	0.95372
50	0.61575	0.023360	0.64893	0.017125	0.92093
51	0.67379	0.027251	0.69343	0.014546	0.88851
52	0.72109	0.030644	0.73586	0.026446	0.95617
53	0.70945	0.040971	0.72731	0.030913	0.96771
54	0.70169	0.035008	0.72213	0.026234	0.95969
55	0.70879	0.019695	0.72950	0.015061	0.89664
56	0.73565	0.018369	0.75103	0.014516	0.88383
57	0.76906	0.025342	0.77990	0.019949	0.93628
58	0.76424	0.018768	0.77969	0.016089	0.86364
59	0.79261	0.020876	0.80014	0.014511	0.88380
60	0.76211	0.037118	0.77979	0.018762	0.94226
61	0.79526	0.028445	0.80472	0.017620	0.93066
62	0.79611	0.020378	0.80633	0.019128	0.92012
63	0.79848	0.019243	0.80904	0.018881	0.91364
64	0.77882	0.028785	0.79560	0.020677	0.94336
65	0.80050	0.022536	0.81248	0.014697	0.89358
66	0.81048	0.017891	0.82193	0.014580	0.87943
67	0.83722	0.030481	0.84473	0.019351	0.94067
68	0.82284	0.017670	0.83670	0.015098	0.87516
69	0.83656	0.036960	0.84988	0.036601	0.96892
70	0.87306	0.060648	0.88042	0.052145	0.98035
71	0.89163	0.053742	0.89869	0.057303	0.97988
72	0.91685	0.066257	0.92108	0.068482	0.98276
73	0.97218	0.092037	0.96556	0.078673	0.98505
74	1.11855	0.16182	1.07977	0.10141	0.98700
75	1.40859	0.29831	1.31299	0.18695	0.98819
76	1.52625	0.14099	1.41557	0.10390	0.98687
77	1.58924	0.089707	1.46911	0.066688	0.98424
78	1.68798	0.13153	1.55097	0.097972	0.98662
79	1.80907	0.15784	1.65079	0.11752	0.98725
80	2.12302	0.33058	1.90002	0.20647	0.98828
81	2.50199	0.35059	2.19941	0.17693	0.98820

Table 3: Parameter Estimates of the (Inverse) Indirect Utility Function

Parameter	Estimate	t-Statistic
α_k	.083986	19.0152
α_{kk}	.084275	9.82739
α_{kl}	.026041	2.35158
α_{km}	-.222327E-02	-.136395
α_{ky}	-.035355	-1.10671
α_{kr}	-.022927	-.834667
$\alpha_{k\sigma_m}$	.222818E-02	1.38918
$\alpha_{k\sigma_r}$	-.145590E-02	-.948150
$\alpha_{k\sigma_{rm}}$	.010210	3.88024
α_{lr}	-.382418	-2.34022
α_{mr}	-1.40768	-3.33357
α_{ry}	-.367786	-.483285
α_{rr}	.990246	1.47005
α_{rt}	-.970220E-02	-4.00083
α_l	.344169	20.0940
α_{ll}	.246785	4.14650
α_{lm}	.264223	2.68091
α_{ly}	.258439	1.45899
$\alpha_{l\sigma_m}$	.701642E-02	1.73657
$\alpha_{l\sigma_r}$	-.570200E-02	-1.43794
$\alpha_{l\sigma_{rm}}$	.021931	3.81539
α_m	.729908	19.9491
α_{mm}	1.66873	5.88237
α_{my}	1.04358	2.23935
$\alpha_{m\sigma_m}$	-.441275E-02	-.884382
$\alpha_{m\sigma_r}$	.830805E-02	1.58899
$\alpha_{m\sigma_{rm}}$	-.921800E-03	-.125081
Equation	R^2	D.W.-Statistic
Capital	.855	1.725
Labour	.827	1.994
Materials	.971	1.927

Table 4: Elasticities Evaluated at the Point of Approximation

Elasticity	Estimate	t-Statistic
μ_{kw_k}	-0.019	-0.299
μ_{lw_l}	-0.665	-6.059
$\mu_{m\bar{w}_m}$	-0.121	-0.934
μ_{kw_l}	-0.072	-0.398
μ_{lw_k}	0.052	1.948
$\mu_{k\bar{w_m}}$	-0.434	-3.996
μ_{mw_k}	-0.260	-1.875
$\mu_{l\bar{w}_m}$	-0.634	-2.824
μ_{mw_l}	-0.020	-0.367
$\mu_{k\sigma_{rm}}$	0.122	3.949
$\mu_{l\sigma_{rm}}$	0.064	3.802
$\mu_{m\sigma_{rm}}$	-0.001	-0.125
$\mu_{k\sigma_m}$	0.026	1.383
$\mu_{l\sigma_m}$	0.020	1.729
$\mu_{m\sigma_m}$	-0.006	-0.891
$\mu_{k\sigma_r}$	-0.017	-0.943
$\mu_{l\sigma_r}$	-0.016	-1.434
$\mu_{m\sigma_r}$	0.113	1.623