Product Differentiation under Uncertainty

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Abstract

In this paper we consider the effects of uncertainty on product differentiation by two oligopolistic firms within the context of the standard Hotelling model. We examine a subgame perfect equilibrium of a two-stage non-cooperative game. In the first stage, firms choose their location before market conditions (location) are known. In the second stage, once uncertainty is resolved, they compete in prices. We show that for levels of uncertainty which are not "too high", a unique pure-strategy Nash equilibrium of the two stage game exists.

We show that the degree of product differentiation will be higher under uncertainty, and will increase with uncertainty. Furthermore, for low level of uncertainty, product differentiation is "extreme", whereas for higher levels of uncertainty, differentiation is less than extreme (but still higher than under certainty and increasing with uncertainty).

Keywords: Product Differentiation, Uncetainty, Two stage game.

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1 Introduction

Starting with the seminal paper by Hotelling (1927), product differentiation has been the subject of numerous articles in which its role in determining price competition, entry deterrence, product selection, etc. is examined[1]. One of the questions addressed in the literature is the degree to which product differentiation will be exercised and consequently, the degree to which it will tend to increase price competition. In their important paper, d'Aspremont, et. al. (1979), show that in a market in which firms compete in prices and locations (product choices), product differentiation will be maximal, hence reducing price competition. De Palma et. al. (1985) and Economides (1986), on the other hand, show that under certain cost and preferences conditions, it is possible to get less than maximal, or even minimal product differentiation. Clearly, the optimal degree of product differentiation will be determined by the trade-off between the desire to reduce price competition (by greater differentiation) and the need to be "near" the market.

In general, however, information on market conditions is imperfect. For example, demand conditions are usually unknown, (at least initially) due to imperfect information on the distribution of consumers (or uncertainty regarding any of the other variables/parameters that determine demand functions). How does this affect the trade-off between the desire to reduce price competition and the need to be "near" the market? Specifically, will product differentiation be smaller under uncertainty? Will an increase in uncertainty increase or decrease price competition? Intuitively, it would seem that uncertainty about the "location" of the market, would tend to make firms more conservative, in the sense that they will move closer together, in order not to "miss" the market.

The purpose of this paper is to examine product differentiation under uncertain market conditions. Specifically, we consider a model in which two firms choose both prices and locations, given an uncertainty distribution of consumers, characterized by unknown "end points". The firms engage in a two-stage non-cooperative game with imperfect information. We assume that the firms have to choose their locations, in the first stage, before the location of the market (the location of its "end points") is known. In the second stage, once uncertainty is resolved and given the choice of locations, the firms play a Bertrand-Nash game in which prices are determined. We examine a subgame perfect Nash equilibrium of this two-stage game.

We show that for levels of uncertainty which are not "too high", a unique pure-strategy Nash equilibrium of the two stage game exists. We also show that the degree of product differentiation will be higher under uncertainty (compared with the certainty case) and moreover, an increase in uncertainty will increase the degree of product differentiation. Depending on the level of uncertainty, there will be two types of equilibria. For low level of uncertainty, product differentiation is "extreme", in the sense that it equals the distance between

[1] See for example d'Aspremont, et. al. (1979), de Palma, et.al., (1985), Bonano, G., (1987), Dasgupta, P. and E. Maskin, (1986), Donnenfeld and Weber, (1992). See also Tirole (1988), for a general discussion.

the two most extreme points of the distance distribution. For higher levels of uncertainty, differentiation is less than extreme, but still higher than under certainty (and increasing with uncertainty). Thus, the risk of "missing the market" does not lead the firms to move away from the corners. Intuitively, this is, due to the fact that, although the firms are risk neutral, the-two stage decision process (game) introduces convexity (in random variables), hence risk affinity, into the firms' (second period) profits. Consequently, (increased) uncertainty does not lead the firms to reduce risk taking by moving toward the centre. The reduction in price competition, due to increased differentiation, still has a dominant effect in the trade-off, thus leading the firms to move even further apart.

2 The Model

We consider the Hotelling model, as developed in d'Aspermont, et., al, (1979). There are two duopolists who produce an identical product at constant marginal cost, c. The firms face a continuum of consumers that are distributed (either physically, or in terms of their taste "location") uniformly along the interval $[\gamma, \theta]$. We assume that initially the firms do not know the location of this interval. Specifically, γ and θ are random variables which are assumed to be distributed according to the uniform distribution, g :

$$g(\gamma,\theta) = \begin{cases} [(\overline{\gamma}-\underline{\gamma})(\overline{\theta}-\underline{\theta})]^{-1} & if \quad \underline{\gamma} \leq \gamma \leq \overline{\gamma},\ \underline{\theta} \leq \theta \leq \overline{\theta} \\ 0 & otherwise \end{cases} \tag{1}$$

where $\overline{\gamma} > \underline{\gamma}, \overline{\theta} > \underline{\theta}$. For simplicity, we assume that the intervals, $[\underline{\gamma}, \overline{\gamma}]$, $[\underline{\theta}, \overline{\theta}]$ do not overlap and have the same length[2], Thus, $\overline{\gamma} < \underline{\theta}$ and $\overline{\gamma} - \underline{\gamma} = \overline{\theta} - \underline{\theta} \equiv h \equiv 2d$. We normalize the distance between the means to be unity and the mean of γ to be 1/2, so that $E(\theta) - E(\gamma) = 3/2 - 1/2 = 1$. Given this normalization, require that $0 < h < 1$. Following d'Aspremont, et. al. (1979), we assume that consumers face a quadratic "transportation" cost function, so that a consumer travelling a distance of $x,$ incurs costs of $\mathcal{T} = tx^2,\ t > 0$.

The firms engage in a two-stage non-cooperative game with imperfect information[3]. We assume that they have to choose their locations, in the first stage, before demand conditions are known. This choice is assumed to be irreversible (hence credible). In the second stage, once uncertainty is resolved and given their choice of locations, the firms play a Bertrand-Nash game in which prices are determined. The outcome of the second stage game (including its dependence on first period decisions) is taken into account by the firms when they make their location decisions. Thus, we examine a subgame perfect Nash equilibrium of this two-stage game.

[2] The case of unequal lengths will be discussed below. The fact the there is no overlap, simplifies the integrals in the first stage of the problem.

[3] Similar two stage games under uncertainty are examined in Appelbaum and Lim (1985) and Appelbaum and Weber (1993), (1994), within a different context.

2.1 The Second Stage Bertrand-Nash Game:

Consider the second stage of the game. Let the firms' locations be given by a and b, where, without loss of generality, we assume that: $a \leq b$. We will refer to the firm on the left as firm 1. Given the choice of locations, a and b, and given the information on the location of consumers, i.e., given the values of γ and θ, the demand curves for the two firms are given by

$$D^1(p_1, p_{2,}, a, b, \gamma, \theta) = \frac{x^0 - \gamma}{\theta - \gamma} \tag{2}$$

$$D^2(p_1, p_{2,}, a, b, \gamma, \theta) = \frac{\theta - x^0}{\theta - \gamma} \tag{3}$$

where the "marginal consumer", x^0, is implicitly given by the equation[4],

$$p_1 + t(x^0 - a)^2 = p_2 + t(b - x^0)^2 \tag{4}$$

assuming that $\gamma \leq x^0 \leq \theta$[5] (and assuming that the net surplus that consumers derive from the product is sufficiently high, so that the whole market is covered). The two firms' profit are, therefore, given by

$$\Pi_1(p_1, p_{2,}, a, b, \gamma, \theta) = (p_1 - c)\left[\frac{x^0 - \gamma}{\theta - \gamma}\right] \tag{5}$$

$$\Pi_2(p_1, p_{2,}, a, b, \gamma, \theta) = (p_2 - c)\left[\frac{\theta - x^0}{\theta - \gamma}\right] \tag{6}$$

The second stage Nash equilibrium prices, $p_1(a, b, \gamma, \theta)$, $p_2(a, b, \gamma, \theta)$, are given by the solution to the problems:

$$\max_{p_1} \ \Pi_1(p_1, p_{2,}, a, b, \gamma, \theta) \equiv J_1(a, b, \gamma, \theta) \tag{7}$$

$$\max_{p_2} \ \Pi_2(p_1, p_{2,}, a, b, \gamma, \theta) \equiv J_1(a, b, \gamma, \theta). \tag{8}$$

PROPOSITION 1: For all $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$, $\underline{\theta} \leq \theta \leq \overline{\theta}$ and locations a and b, such that $E(\gamma) - h/2 \leq a \leq E(\theta) + h/2$, $E(\gamma) - h/2 \leq b \leq E(\theta) + h/2$, there is a value, $h^* > 0$, such that for all $h < h^*$, there exists a unique Nash equilibrium in prices with: $p_1(a, b, \gamma, \theta) > c$, $p_2(a, b, \gamma, \theta) > c$.

Proof: Given the profit functions in (5) and (6), the two reaction functions are given by the linear functions:

$$p_1 = \frac{c + p_2 + t(b - a)(a + b - 2\gamma)}{2} \tag{9}$$

$$p_2 = \frac{c + p_1 + t(b - a)(2\theta - a - b)}{2} \tag{10}$$

Subtracting the (horizontal and vertical, respectively) intercepts of the reaction functions in (9) and (10), we get $p_1(0) - p_2^{-1}(0) = .5[3c + t(b - a)(4\theta - 2\gamma - a - b)]$ and $p_2(0) - p_1^{-1}(0) = .5[3c + t(b - a)(2\theta - 4\gamma + a + b)]$.

[4] Thus the marginal consumer is given by: $x^0 = \frac{(b+a)}{2} + \frac{(p_2 - p_1)}{2t(b-a)}$.

[5] The condition that guarantees that $\gamma \leq x^0 \leq \theta$ is that markups are positive. A sufficient condition for this is provided in proposition 1 below.

Furthermore, $\frac{\partial p_2(\cdot)}{\partial p_1} = .5 < \frac{\partial p_1^{-1}(\cdot)}{\partial p_1} = 2$, i.e., $p_1\ (\cdot)$ is steeper than $p_2(\cdot)$ (with respect to the horizontal axis). Since the reaction functions are linear, if it can be shown that $p_1(0) - p_2^{-1}(0) > 0$ (or that $p_2(0) - p_1^{-1}(0) > 0$) then, there must be a unique solution. Solving the two equations in (9) and (10) we obtain the Nash equilibrium prices as:

$$p_1 = c + \frac{t(b-a)(2\theta - 4\gamma + a + b)}{3} \tag{11}$$

$$p_2 = c + \frac{t(b-a)(4\theta - 2\gamma - a - b)}{3} \tag{12}$$

A necessary and sufficient condition for the markups, $p_1 - c$, and $p_2 - c$, to be positive (since $b \geq a$) is that $2\theta - 4\gamma + a + b > 0$ and $4\theta - 2\gamma - a - b > 0$. Note that this will also ensure that the intercept conditions above are satisfied. Now, defining the sum of a and b as $a + b \equiv S$, the condition for positive markups can be written as:

$$S > 4\gamma - 2\theta, \qquad S < 4\theta - 2\gamma \tag{13}$$

But: $S - (4\gamma - 2\theta) > Min\, S - \ Max\ (4\gamma - 2\theta) = 2(.5 - h/2) - (4\overline{\gamma} - 2\underline{\theta}) = (1-h) - (3h-1) = 4(1/2 - h)$, and $S - (4\theta - 2\gamma) <\ Max\ S\ - Min\ (4\theta - 2\gamma) =\ 2(1.5 + h/2) - (4\underline{\theta} - 2\overline{\gamma}) = (3+h) - ((5-3h) = 4(h - 1/2)$. Hence a sufficient condition for the existence of a unique solution with $p_1 - c > 0$, and $p_2 - c > 0$, is that $h < h* = 1/2$. We can weaken this condition by recognizing that the problem is symmetric (in the sense that the two firms are identical and the distributions of γ, and θ are symmetric around $E(\gamma) = .5$ and $E(\theta) = 1.5$), so that it must be the case the solution will be symmetric. In other words, whatever the location choices are, they must be such that they are equally distant from the midpoint between the means of γ and θ. Since we took this midpoint to be one, it must be the case that $b - 1 = 1 - a$, so that, $a + b = 2$. Given this symmetry, conditions (13), are satisfied for all, (γ, θ), if $h < 1$. Hence a sufficient condition for the existence of a unique symmetric Nash equilibrium in prices is that $h < 1$[6]. □

2.2 The First Stage Cournot-Nash Game:

The two firms play a Cournot-Nash game in the first stage. We assume that the firms are risk neutral, so that each firm chooses its location to maximize the expected value of J^i, given the other firm's of location. Given the distribution of γ and θ and using the second stage equilibrium prices in (11) and (12), the firms' expected profits can be written as:

$$V_1(a,b,h) \equiv E[J_1(a,b,\gamma,\theta)] = \frac{t}{18h^2} \int_{\underline{\gamma}}^{\overline{\gamma}} \int_{\underline{\theta}}^{\overline{\theta}} \frac{(b-a)(a+b+2\theta-4\gamma)^2}{\theta - \gamma}\ d\theta\, d\gamma \tag{14}$$

$$V_2(a,b,h) \equiv E[J_2(a,b,\gamma,\theta)] = \frac{t}{18h^2} \int_{\underline{\gamma}}^{\overline{\gamma}} \int_{\underline{\theta}}^{\overline{\theta}} \frac{(b-a)(4\theta-2\gamma-a-b)^2}{\theta - \gamma}\ d\theta\, d\gamma \tag{15}$$

[6]Note that this also guarantees that $\gamma \leq x^0 \leq \theta$, as was pointed out in footnote 5 above.

PROPOSITION 2: For all $0 < h < 1$, and for all $0 \leq a \leq 1, 1 \leq b \leq 2$, the expected values of the second stage maximum profits functions, $V_1(a,b,h)$, and $V_2(a,b,h)$, are continuous and strictly concave in a, b, respectively.

Proof: Differentiating (14) and (15) we get: $\partial^2 V_1/\partial a^2 = -\frac{t}{18h^2}[12h^2 + 2(3a+b-4)[(1+h)\ln(1+h) + (1-h)\ln(1-h)]]$, and $\partial^2 V_2/\partial b^2 = -\frac{t}{18h^2}[12h^2 + 2(4-a-3b)[(1+h)\ln(1+h) + (1-h)\ln(1-h)]]$, which are strictly negative for all, $0 \leq a \leq 1$, $1 \leq b \leq 2$, $0 < h < 1$[7]. $\square$

For example, the expected profit function, $V_1(a,b,h)$ is shown in Figure 1, for the case when $h = .8$[8]. As the Figure 1 shows (best seen from the contours of $V_1(a,b,h)$), $V_1(a,b,h)$ is indeed concave with respect to a ($V_2(a,b,h)$ looks the same with respect to the b axis).

The two firms solve the following problems

$$\max_a \,[\, V_1(a,b,h) : a \geq E(\gamma) - h/2, \text{ given } b] \tag{16}$$

$$\max_b \,[\, V_2(a,b,h) : b \leq E(\theta) + h/2, \text{ given } a] \tag{17}$$

which yield the two Kuhn-Tucker conditions:

$$\frac{\partial V_1(a,b,h)}{\partial a} - \lambda_1 \leq 0, \; \left(\frac{\partial V_1(a,b,h)}{\partial a} - \lambda_1\right) a = 0, \; a \geq 0 \tag{18}$$

$$a - E(\gamma) + h/2 \geq 0, \; (a - E(\gamma) + h/2)\,\lambda_1 = 0, \; \lambda_1 \geq 0 \tag{19}$$

$$\frac{\partial V_2(a,b,h)}{\partial b} - \lambda_2 \leq 0, \; \left(\frac{\partial V_2(a,b,h)}{\partial b} - \lambda_2\right) b = 0, \; b \geq 0 \tag{20}$$

$$E(\theta) + h/2 - b \geq 0, \; (E(\theta) + h/2 - b)\,\lambda_2 = 0, \; \lambda_2 \geq 0 \tag{21}$$

where λ_1 and λ_2 are the Lagrangean multipliers corresponding to the two constraints, and

$$\frac{\partial V_1(a,b,h)}{\partial a} = \frac{t}{18h^2}\int_{\underline{\gamma}}^{\overline{\gamma}}\int_{\underline{\theta}}^{\overline{\theta}} \frac{(a+b+2\theta-4\gamma)(b-3a-2\theta+4\gamma)}{\theta-\gamma}\, d\theta\, d\gamma \tag{22}$$

$$\frac{\partial V_2(a,b,h)}{\partial b} = \frac{t}{18h^2}\int_{\underline{\gamma}}^{\overline{\gamma}}\int_{\underline{\theta}}^{\overline{\theta}} \frac{(4\theta-2\gamma-a-b)(a-3b+4\theta-2\gamma)}{\theta-\gamma}\, d\theta\, d\gamma. \tag{23}$$

The Kuhn-Tucker condition (19) implies that if $a > E(\gamma) - h/2$, then we must have $\lambda_1 = 0$, so that in (18) we must have $\frac{\partial V_1(a,b,h)}{\partial a} = 0$, i.e., we have an interior solution for a. Similarly The Kuhn-tucker conditions (21) implies that if $b < E(\theta) + h/2$, then we must have $\lambda_2 = 0$, so that in (20) we must have $\frac{\partial V_2(a,b,h)}{\partial b} = 0$, that is, we have an interior solution for b. On the other hand, if $\lambda_1 > 0$, then $a = E(\gamma) - h/2 > 0$, so that in (18) we must have $\frac{\partial V_1(a,b,h)}{\partial a} - \lambda_1 = 0$, i.e., $\frac{\partial V_1(a,b,h)}{\partial a} < 0$, and we have a corner solution for a. Similarly, if $\lambda_2 > 0$, then $b = E(\theta) + h/2$, so that in (20) we must have $\frac{\partial V_2(a,b,h)}{\partial b} - \lambda_2 = 0$, i.e., $\frac{\partial V_2(a,b,h)}{\partial b} < 0$, and we have a corner solution for b.

[7] Furthermore, the determinants of the Hessian matrices are strictly positive.

[8] All the figures in this paper were drawn using the programme Maple V for Windows.

Given the strict concavity of V_1 and V_2, we can define the two reaction functions: $a = R_1(b,h)$ and $b = R_2(a,h)$, from the conditions $\frac{\partial V_1(a,b,h)}{\partial a} = 0$ and $\frac{\partial V_2(a,b,h)}{\partial b} = 0$, respectively. Hence, for different values of h, we will have different reaction functions. For example, Figure 2 shows the reaction functions for the two cases when $h = .8$ (thin lines), and $h = .2$ (thick lines). The steeper curves in Figure 1 are the reaction functions for Firm 1 (for the two cases; $h = .8$ and $h = .2$) and the downward sloping line, $a + b = 2$, is the symmetry line. What can be said about the nature of the solution?

PROPOSITION 3: For $0 < a < 2,\ 0 < b < 2$, there exists a symmetric and unique intersection of the two reaction functions, $(a^\star, b^\star)$, where, $a^\star = R_1(b^\star, h)$ and $b^\star = R_2(a^\star, h)$.

Proof: Given the concavity of the V_1 and V_2, the reaction functions are continuous. To make sure that they intersect we check their intercepts. From (22) and (23) we get $R_2(0,h) > R_1^{-1}(0,h)$ and $R_1(2,h) < R_2^{-1}(2,h)$, for all $0 < h < 1$[9]. Thus, the two reaction functions must have an intersection for $0 < a < 2,\ 0 < b < 2$. To show that the this intersection is unique we have to compare the slopes $\partial R_1^{-1}(a,h)/\partial a$ and $\partial R_2(a,h)/\partial a$. Calculating these slopes we get that $\partial R_1^{-1}(a,h)/\partial a > \partial R_2(a,h)/\partial a$, for all $0 < h < 1$. Thus, over the range, $0 < a < 2,\ 0 < b < 2$, the intersection of the two reaction functions is unique[10]. Since the two firms are identical and the distributions of γ, and θ are symmetric around $E(\gamma) = .5$ and $E(\theta) = 1.5$, it must be the case the solution for $a^\star, b^\star$ will be symmetric. In other words, they will be equally distant from the $[E(\gamma) + E(\theta)]/2 = 1$. Thus, $1 - a^\star = b^\star - 1$ so that, $a^\star + b^\star = 2$, as can be seen in Figure 2. Using (22) and (23) we can solve the equations $\frac{\partial V_1(a,b,h)}{\partial a} = 0$ and $\frac{\partial V_2(a,b,h)}{\partial b} = 0$, explicitly to obtain the intersection of the two reaction functions as:

$$a^\star = \frac{5}{6}\left\{\frac{14}{3} - [\ln(1+h) + \ln(1-h)](1 + \frac{1}{3h^2}) + [\ln(1+h) - \ln(1-h)](\frac{1}{3} - \frac{1}{h})\right\} \tag{24}$$

$$b^\star = 2 - a^\star \Box \tag{25}$$

Figure 3 shows the solution (unconstrained) $a^\star$, as a function of $d \equiv h/2$. As Figure 3 indicates (and as can be verified from (24), $a^\star$ is a decreasing function of h, for all $0 < h < 1$. Since $b^\star = 2 - a^\star$, this implies that $b^\star$ is an increasing function of h.

Now, the question is whether we have interior, or corner solutions.

PROPOSITION 4: There exists a value h^+, $0 < h^+ < 1$, such that for all $h \leq h^+$, (and all $0 < a < 2,\ 0 < b < 2$) we have $\frac{\partial V_1(\underline{\gamma}, b^\star, h)}{\partial a} < 0$, and $\frac{\partial V_2(a^\star, \overline{\theta}, h)}{\partial b} < 0$ and for all $h \geq h^+$, we have $\frac{\partial V_1(a^\star, b^\star, h)}{\partial a} = 0$, and $\frac{\partial V_2(a^\star, b^\star, \gamma)}{\partial b} = 0$.

Proof: Proposition 4, says that for all $h \leq h^+$, we have a corner solution, with $a = E(\gamma) - h/2 > 0$, and $b = E(\theta) + h/2 < 2$ and for all $h \geq h^+$, we have an interior solution, with $a = a^\star$, and $b = b^\star$. Using the optimal

[9] See, for example Figures 4_a and 4_b.

[10] See Figures 4_a and 4_b. It should be pointed out that the two reaction functions have other intersections, but these are for values of a, b, which do not satisfy: $0 < a < 2,\ 0 < b < 2$. At these other intersections the second order conditions for a maximum are not satisfied.

value for $a^\star$ in (24), we solve the equation

$$\frac{5}{6}\left\{\frac{14}{3}-[\ln(1+h)+\ln(1-h)](1+\frac{1}{3h^2})+\right. \tag{26}$$

$$\left.[\ln(1+h)-\ln(1-h)](\frac{1}{3}-\frac{1}{h})\right\}=1/2-h/2$$

and obtain the value of $h^+ = .5072$. Thus, for all for all $h \leq h^+$, we have a corner solution, with $a = E(\gamma) - h/2 > 0$, and $b = E(\theta) + h/2 < 2$ and for all $h \geq h^+$, we have an interior solution, with $a = a^\star$, and $b = b^\star$[11]. □

If we now define the solution to the Kuhn-Tucker conditions (18)-(21) by $(a^0, b^0, \lambda_1^0, \lambda_2^0)$, then we have that

$$\begin{array}{c} a^0 = a^\star,\ \ b^0 = b^\star,\ \ \lambda_1^0 = 0,\ \ \lambda_2^0 = 0 \quad \text{for all} \quad h \geq h^+ \\ a^0 = E(\gamma) - h/2,\ \ b = E(\theta) + h/2,\ \ \lambda_1^0 > 0,\ \ \lambda_2^0 > 0 \quad \text{for all} \quad h \leq h^+ \square \end{array} \tag{27}$$

The vector $\{p_1(a^0, b^0, \gamma, \theta),\ p_2(a^0, b^0, \gamma, \theta), a^0, b^0\}$, constitutes a subgame perfect equilibrium. This is shown in Figure 4, where a^0, b^0 are given by the thick lines (where the shallow curves in the in Figure 4 are the $a^\star$, $b^\star$ curves and the steeper lines are the constraints, $E(\gamma) - h/2$, and $E(\theta) + h/2$). As we can see in the diagram, for all $d = h/2 < .2536$, we have a corner solution and for all $d = h/2 > .2536$, we have an interior solution.

As for the effects of increased uncertainty on the degree of product differentiation, we have the following result:

PROPOSITION 5: (i) For any $0 < h < 1$, the degree of product differentiation will be higher under uncertainty (compared with the case when $\gamma = E(\gamma)$ and $\theta = E(\theta)$, with certainty), (ii) an increase in uncertainty will increase the degree of product differentiation.

Proof: Since a^* and $E(\gamma) - h/2$ are decreasing functions of h,whereas, b^* and $E(\gamma) - h/2$ are increasing function of h, it follows that $\partial(b^0 - a^0)/\partial h > 0$. The solution in the certainty case when $\gamma = E(\gamma)$ and $\theta = E(\theta)$, is given by the limiting case when $h \to 0$. In this case we get $a^\star = E(\gamma) = .5$, $b^\star = E(\theta) = 1.5$, which in the standard result[12]. For any $0 < h < 1$, we will, therefore, have greater differentiation under uncertainty. Hence, both the impact effect and the marginal effect of uncertainty are to increase differentiation. Proposition 5 is demonstrated in Figure 4.□

Proposition 5, implies that product differentiation increases with uncertainty. However, whereas for low level of uncertainty (when $h \leq h^+$), product differentiation is "extremely" maximal, in the sense that it equals $\overline{\theta} - \overline{\gamma} = 1 + h$, for higher levels of uncertainty, it is less than the extreme possible value, with $b^\star - a^\star < 1 + h$. Although it may seem that the greater risk that is associated with higher uncertainty, would lead the firms to move away from the corners in order not to "miss the market", our result, shows that the reduction in price competition, due to increased differentiation, will have a dominant effect, leading the firms to move even further apart.

[11] For the special case when $h = h^+ = .5072$, we have $a^* = .2464 = E(\gamma) - h/2 = .5 - .2536$.
[12] See for example, d'Aspremont et. al., (1979), Tirole (1989).

The reason we obtain this result is that the two stage nature of the game (decision), introduces a convexity into the firms' second period maximum profits. Thus we have:

PROPOSITION 6: The second period maximum profits; $J_1(a, b, \gamma, \theta)$, and $J_2(a, b, \gamma, \theta)$, are convex in γ, θ, for all, $0 < \gamma < 1,\ \ 1 < \theta < 2,\ 0 < a < 1,\ \ 1 < b < 2$.

Proof: If we define the Hessian matrix for J_i as, $[\mathcal{A}^i]_{\gamma\theta}$, then using (14) and (15) we can write it as:

$$[\mathcal{A}^i]_{\gamma\theta} = \begin{pmatrix} 2\frac{(a+b-2\gamma)^2}{(\theta-\gamma)^3}, & 2\frac{\{(a+b)[2(\theta+\gamma)-(a+b)]-4\theta\gamma\}}{(\theta-\gamma)^3} \\ 2\frac{\{(a+b)[2(\theta+\gamma)-(a+b)]-4\theta\gamma\}}{(\theta-\gamma)^3}, & 2\frac{(a+b-2\theta)^2}{(\theta-\gamma)^3} \end{pmatrix} \tag{28}$$

which is positive semi-definite with strictly positive diagonal terms for all $0 < \gamma < 1,\ \ 1 < \theta < 2,\ 0 < a < 1,\ \ 1 < b < 2$.□

It should be noted that this is a common result in problems of choice under uncertainty[13] and is similar to the standard result in duality theory, that profit functions are convex in prices[14]. For example, a competitive firm that chooses its capital before it knows the price of output and than chooses output, after observing the price, will have second period profits which are convex in price. Depending on the shape of its cost function, such a firm may actually increase its investment if uncertainty increases. Essentially, we have a similar result here. Although the firms are risk neutral, the resolution of uncertainty in this two-stage game introduces risk affinity into their profit functions. Consequently, (increased) uncertainty does not lead the firms to behave more conservatively (by decreasing product differentiation). In fact, due to the risk affinity, the effect of uncertainty is the opposite; the firms increase product differentiation.

It is, of course possible to introduce risk aversion into the model. Presumably, risk aversion will have the opposite effect and will mitigate the benefits from increased differentiation. With risk aversion it is, therefore, possible that the net effect will be reduced price competition.

2.3 Asymmetric Spreads:

Before we conclude, it is useful to note that model we considered was perfectly symmetric. The two firms were identical and the uncertainty on both sides of the market was symmetric, in the sense that: $\overline{\gamma} - \underline{\gamma} = \overline{\theta} - \underline{\theta} \equiv h$. Consequently, the solution was symmetric. Without examining an asymmetric model in detail, let us briefly consider the effects of an asymmetry in the distribution of the random variables. Specifically, suppose that one of the corners becomes more uncertain, in the sense that its spread increases. The two expected profit functions (14) and (15) can now be written as:

$$V_1(a, b, h_1, h_2) = \frac{t}{18h_1^2h_2^2} \int_{..5-h_1}^{.5+h_1} \int_{1..5-h_2}^{1..5+h_2} \frac{(b-a)(a+b+2\theta-4\gamma)^2}{\theta-\gamma}\, d\theta\, d\gamma \tag{29}$$

$$V_2(a, b, h_1, h_2) = \frac{t}{18h_1^2h_2^2} \int_{..5-h_1}^{.5+h_1} \int_{1..5-h_2}^{1..5+h_2} \frac{(b-a)(4\theta-2\gamma-a-b)^2}{\theta-\gamma}\, d\theta\, d\gamma$$

[13] See for example, Hartman (1976), Epstein (1978) and Appelbaum and Katz (1986).
[14] See for example Diewert (1982), Varian (1978).

where $h_1 \equiv \overline{\gamma} - \underline{\gamma}$, $h_2 \equiv \overline{\theta} - \underline{\theta}$. The two corresponding reaction functions are given by the equations: $\frac{\partial V_1(a,b,h_1,h_2)}{\partial a} = 0$ and $\frac{\partial V_2(a,b,h_1,h_2)}{\partial b} = 0$. Let us denote them by : $a = R_1(b, h_1, h_2)$ and $b = R_2(a, h_1, h_2)$, respectively. It can be shown that $\frac{\partial R_1(b,h_1,h_2)}{\partial h_1} < 0$, $\frac{\partial R_1(b,h_1,h_2)}{\partial h_2} > 0$, $\frac{\partial R_2(b,h_1,h_2)}{\partial h_1} < 0$ and $\frac{\partial R_2(b,h_1,h_2)}{\partial h_2} > 0$. An example is demonstrated in Figure 5, where we set the spreads initially to be $h_1 = h_2 = .6$ (which ensures an initial interior solution for both firms) and then we let h_2 increase to $h_2 = .8$, with h_1 remaining constant at $h_1 = .6$. The thick curves in Figure 5, correspond to the higher h_2, and the thin ones, are the original curves. For convenience we also added the line $a + b = 2$, representing symmetric solutions (note that the original solution is on this line). As we can see in the diagram, as a result of the increase in h_2, the reaction function of firm 1 (the steeper one) has rotated to the right, whereas the reaction function of firm 2 rotated upward to the left. Consequently, both a and b must increase. But this implies that firm 1 moves closer to the centre, whereas firm 2 moves further away, which means that both firms tend to move toward "the more uncertain corner". Although this seem a surprising result, it may be understood in view of the convexity of the second period profit functions in the random variables. Given this convexity, the dominant motive is to reduce price competition by increasing the distance. We can see this by noting that since firm 2 is assumed to be on the right, the effect of an increase in h_2 on its reaction function is greater than the effect on firm 1's reaction function. Therefore, while both a and b increase, b increases more than a, so that distance between the two firms increases and we get greater product differentiation.

We can calculate the solutions, before and after the increase in h_2 and find that when $h_1 = h_2 = .6$, we get $a^\star = .2448$, $b^\star = 1.7551$, and with $h_1 = .6$, $h_2 = .8$, the solution is: $a^\star = .25848$, $b^\star = 1.7735$. Note that since $a^\star = .25848 > .5 - h_1/2 = .2$, $b^\star < 1.5 + h_2/2 = 1.9$, we have an interior solution, $a^0 = a^\star$, $b^0 = b^\star$, for both firms. Calculating the distance between the two firms, we get: $b^0 - a^0 = 1.7735 - .25848 = 1.515$ which is greater than the initial distance. An asymmetric increase in the spread will, therefore, increase product differentiation. The following is example of the effects of an asymmetric change in spreads, when there is a corner solution:

1. $h_1 = .02$, $h_2 = .98$, $a^\star = .293 < .5 - h_1/2 = .49$, $b^\star = 1.814 < 1.5 + h_2/2 = 1.99$, thus we have a corner solution with $a^0 = .5 - .01 = .49$, for firm 1 and $a^\star = .186 = .49$, for firm 2. The distance between the two firms is: $b^0 - a^0 = 1.324$.

2. The solution is simply reversed if $h_2 = .02$, $h_1 = .98$. We now have: $a^\star = .186 > .5 - h_1/2 = .01$, $b^\star = 1.707 > 1.5 + h_2/2 = 1.51$, thus we have a corner solution with $b^0 = 1.5 + .01 = 1.51$, for firm 2 and $a^0 = a^\star = .186$, for firm 1. Again the distance between the two firms is: $b^0 - a^0 = 1.324$.

Thus, we conclude that an asymmetric change in the spread of the distribution of the unknown corners, will tend to increase product differentiation. Moreover, both firms move toward the more uncertain corner.

3 Conclusion:

In this paper we examined product differentiation under uncertain market conditions. We consider a two-stage non-cooperative game with imperfect information, where firms choose locations in the first stage and then, once uncertainty is resolved, they compete in prices. We examine a subgame perfect Nash equilibrium of this two-stage game and show that for levels of uncertainty which are not "too high", a unique pure-strategy Nash equilibrium of the two stage game exists. We also show that the degree of product differentiation will be higher under uncertainty, and furthermore, it will increase with uncertainty. Depending on the level of uncertainty, there are two types of equilibria. For low level of uncertainty, product differentiation is "extreme" (equals the distance between the two most extreme points of the distance distribution), whereas for higher levels of uncertainty, differentiation is less than extreme, but still higher than under certainty (and still increasing with it). The risk of "missing the market" does not, therefore, lead the firms to move away from the corners. This is due to the convexity (in random variables) which is introduced by the two stage nature of the game. Consequently, (increased) uncertainty does not lead the firms to reduce risk-taking (by moving toward the centre). In the trade-off between the desire to reduce price competition and the need to be "near" the market, the reduction in price competition dominates, thus leading the firms to move even further apart. Finally, we also show that an asymmetric change in the spreads of the distributions of the unknown corners, will tend to increase product differentiation and will lead both firms to move toward "the more uncertain" corner.

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