Estimation of the Saving Function without Expected Utility

Elie Appelbaum* and Parantap Basu†

February 2002

Abstract

In this paper, we use duality properties to show that the saving function can be derived in a non-expected utility framework, even with fairly general preferences. We propose an econometric framework for estimation of the cross-country aggregate saving function, using a flexible functional form for preferences. A set necessary conditions for the validity of expected utility hypothesis emerges, which we test using cross-country savings data. Expected utility hypothesis is rejected by the data. Estimated elasticities of savings with respect to income and interest rate show considerable heterogeneity across rich and poor countries opening up new theoretical issues.

Keywords: Non-Expected utility, Savings, Moments.

Address: Elie Appelbaum, Department of Economics,
York University, 4700 Keele St, Toronto, Ont., M5R 1N5 Canada
phone: 416-736-5083,
e-mail: elie@dept.econ.yorku.ca

*York University and University of Sydney.
†Fordham University.

1 Introduction

In recent years, there has been growing skepticism in the macroeconomics and finance literature about the empirical relevance of the expected utility model (EUM). These increasing doubts emanated from the model's poor performance in experimental tests,[1] the realization that the EUM cannot resolve basic theoretical issues and its inability to explain various aspects of observed behavior. For example, the EUM was not able to explain the equity premium puzzle (Mehra and Prescott (1985), Weil (1989)), or the limited stock market participation of households (Poterba and Samwick (1995)). Similarly, it could not address the difficulties in measuring intertemporal substitution and risk aversion, in the macro literature.[2]

Various non-expected utility models (NEUM) have been proposed in order to deal with the above difficulties. While the use of NEUM addresses some of these difficulties, it may, however, introduce new ones. For example, unlike the EUM, there is no "standard" non-expected utility model. Numerous competing NEUM have been proposed, but it is not always easy to discriminate among them theoretically, or empirically. A further difficulty is that NEUM are difficult to solve explicitly and are, therefore, not easy to apply empirically.

The purpose of this paper is to provide a simple framework for the analysis of general non-expected utility behavior. This framework, based on Appelbaum (1998), is very general in the sense that it is not restricted to any specific model of non-expected utility, namely, it does not assume a specific functional form for the utility function. We demonstrate the framework within the context of household's consumption/saving decisions under uncertainty.[3] Specifically, we show that NEUM household saving functions can be easily obtained by using Roy' identity. Second, we show that standard estimation techniques can then be used to estimate the saving function. Third, we use our framework to derive necessary conditions for the EUM, thus providing a simple direct *econometric* test for expected utility behavior. Finally, to demonstrate the empirical usefulness of our approach, we provide an example of an empirical application. We estimate a cross country aggregate savings function and use it to: (i) test for the validity of the EUM, (ii) estimate elasticities of saving with respect to the first three moments of the distribution of returns and future income and (iii) compare the comparative statics effects of increase in risk of income and interest rates on savings for poor and rich countries. We reject the necessary condition for the EUM (convexity in moments). The EUM must, therefore, be rejected. The estimated elasticities show variation across rich and poor countries, which

[1] See Kahneman and Tversky (1979), Tversky and Kahneman (1986), Machina (1987), Camerer (1989).

[2] Epstein and Zin (1991) examine the possible separation of time/risk preference. Hall (1988) finds elasticity of substitution close to zero using aggregate consumption data. On the other hand, real business cycle and growth literature, calibrates a value of this elasticty close to unity. Gueven (2001) presents a model of limited stock market participation involving a non-expected utlity preference to reconcile these conflcting measures of substitution elasticity.

[3] It can, however, also be applied in a similar fashion in any other context.

is consistent with the existing literature on savings under uncertainty.

The paper is organized as follows. In the following section, we lay out the theoretical framework. We characterize the relationship between moments of the underlying stochastic processes and the indirect utility function. A few illustrative examples, drawn from the existing two period life cycle model, are used to verify the properties of the indirect utility function. In section 3, we discuss the methodology for empirical implementation and in section 4 we discuss the empirical results of the estimation of a cross country aggregate savings function.

2 Theoretical Framework

Consider a standard two-period consumption/saving problem under uncertainty. The individual's current income is y_0 and he expects an uncertain income of p in the next period. The individual faces a risky total return of R. We assume that both R and p are continuous bounded random variable. Let the joint (cumulative) distribution of R and p be given by $G(R,p)$, with the finite support $(R,p) \in \mathcal{A}$. Future consumption is given by:

$$\begin{aligned} c_1 &= p + (y_0 - c_0)R \\ &= p + sR \end{aligned} \tag{1}$$

where c_0, c_1 are present and future consumption choices, respectively and $s = y_0 - c_0$ is saving.

It is useful to write

$$R = \overline{R} + e_r \tag{2}$$

$$p = \overline{p} + e_p \tag{3}$$

where $(e_r, e_p) \sim G_e$ with: $E(e_r) = E(e_p) = 0$, $Var(e_r) = \sigma_r^2$, $Var(e_p) = \sigma_p^2$, $Cov(e_r, e_p) = \sigma_{rp}$ and the distributions G_e can be derived from G. Given the uncertainty with respect to R and p, consumption bundles: $c \equiv (c_0, c_1)$ constitute lotteries. Using the joint distribution of R and p (or using G_e) we can obtain the (cumulative) distribution of c as G_c.[4] It is well known that if the individual's preferences over these consumption lotteries are rational and continuous, the individual's preferences can be represented by a continuous utility function $\mathcal{U} : G_c \to R$ such that: $G_{c1} \succsim G_{c2} \Longleftrightarrow \mathcal{U}(G_{c1}) \geq \mathcal{U}(G_{c2})$.[5]

We can, therefore, write the individual's problem as:

$$\begin{aligned} Max_{c_0} \, \{\mathcal{U}[G_c] \quad &: \quad c_1 = p + (y_0 - c_0)R, \\ (R,\ p) \quad &\sim \quad G\} \equiv J(G; y_0) \end{aligned} \tag{4}$$

[4] Note that although we have a joint distribution, $G_c(c_0, c_1)$, c_0 is a degenerate random variable.
[5] See Mas-Collel et. al. (1995).

where J is the indirect utility functional.

Since $c_0 \in \mathcal{C}$, is a non-empty and compact subset in $\mathcal{R}^1_+$, and given that $\mathcal{U}$ is continuous in G_c (where G_c is continuous in c_0) with a compact range, it follows from the Theorem of the Maximum (Berg (1963)), that J is continuous and the optimal solution for c_0 is upper semi-continuous. This, of course, is the standard result from consumer theory.

If we assume that the individual's underlying preferences also satisfy the independence axiom, which is required for the von Neumann-Morgenstern Theorem,[6] then $\mathcal{U}$ takes the form of expected utility; $\mathcal{U}[G_c] = E\{U[c]\}$, where U is a continuous von Neumann-Morgenstern utility function. Problem (4) can then be written as:

$$Max_{c_0}\ \{\int_{R,p\in\mathcal{A}} U[c]dG(R,p) : c_1 = p + (y_0 - c_0)R\} \tag{5}$$

The additional assumption of independence introduces "linearity in probabilities" which enables us to obtain a standard result from duality theory:

Proposition 1: If $\mathcal{U}[G_c] = E\{U[c]\}$ (with expected utility maximization) then J *is convex in* G.[7]

Proof: See Appendix.

Proposition 2: If $\mathcal{U}[G_c] \neq E\{U[c]\}$ (without expected utility maximization) then J *is not necessarily convex in* G.

Proof: See Appendix.

2.1 Moments and the Indirect Utility Function

While the indirect functional $J(G; y_0)$ is useful theoretically, it is often analytically intractable and, therefore, not very useful from an *empirical* viewpoint. Thus, instead of looking at $J(G; y_0)$, it is desirable to consider characterizations of indirect preferences which may be easier to apply empirically.

For example, define the moments of the distribution G as m. Since the random variables p, R are concentrated on a compact support, the moments exist and uniquely determine the distribution.[8] For any m there is, therefore, a unique distribution whose

[6] For a discussion of the required conditions see, for example, Kreps (1990), Mas-Collel, et.al. (1995).

[7] Machina (1984) (p. 208, Theorems 2 and 3.) also shows that convexity and continuity completely characterize the functional J, so that any functional with these properties is the indirect utility functional for some preferences.

[8] Bounded support is a sufficient condition for the distribution function to be uniquely determined by the moments. This is the so called "moments problem". See, for example, Wilks (1964), theorem 5.5.1. p. 126, Kendall (1969) Corollary 4.22, p. 110.

moments are given by m. Let this distribution be denoted by G^m. The indirect utility above can then be written as $J(G^m, y_0) \equiv H(m, y_0)$.

Since with expected utility J is convex in G, it follows that:

Theorem 1: If $\mathcal{U}[G_c] = E\{U[c]\}$ (with expected utility maximization) then H *is convex in the moments.*

Proof: See Appendix.

Theorem 2: If $\mathcal{U}[G_c] \neq E\{U[c]\}$ (without expected utility maximization), H *may be either convex, or concave in the moments.*

Proof: See Appendix.

Theorem 1 implies that convexity of H is a necessary (but not sufficient) condition for expected utility maximization. Thus, a rejection of the convexity of H implies that we must reject expected utility maximization. On the other hand, Theorem 2 implies that, without expected utility maximization, H may, or may not, be convex in moments. In other words, it is possible to have convexity in some types of non-expected utility models, but concavity in others.

2.2 Special Cases

2.2.1 The Selden Model

For Selden's (1978) OCE preferences, we have

$$\begin{aligned} J(G; y_0) &= \max_{c_0}\{U(c_0) + U(z) : && (6)\\ Z &= V^{-1}\left[\int V(p + (y_0 - c_0)R)dG(R,p)\right]\} && (7) \end{aligned}$$

where $U'(.) > 0, U''(.) < 0$, $V'(.) > 0, V''(.) < 0$.

Consider the distributions G_1, G_2, G_λ (for R and p) where $G_\lambda = \lambda\ G_1 + (1-\lambda)G_2$. Let the corresponding solutions for c_0 be given by: c_0^1 , c_0^2 and c_0^λ respectively. The corresponding solutions for z are z^1, z^2 and z^λ respectively.

For Selden's OCE form we have;

$$J(G_\lambda; y_0) = U(c_0^\lambda) + U(z^\lambda) \tag{8}$$

$$J(G_1; y_0) = U(c_0^1) + U(z^1) \tag{9}$$

$$J(G_2; y_0) = U(c_0^2) + U(z^2) \tag{10}$$

Note that $J(G; y_0)$ is convex in G iff

$$J(G_\lambda, y_0) \leq \lambda J(G_1; y_0) + (1-\lambda) J(G_2; y_0) \tag{11}$$

Can we say anything about the convexity of $J(G; y_0)$? Since $E[V(c_1)]$ is linear in probabilities and $V(z)$ is concave, it follows that $z = V^{-1}\{E[V(c_1)\}$ is convex in probabilities. Hence, since $U(z)$ is concave, we do not know if the (indirect utility functional) is convex, or concave in probabilities. Condition (11) may, therefore, not hold: in general, $J(G)$ may be either convex or concave in G.

Two special cases are of interest:

(i) CES-CRRA Case (Selden,1978):

Assume that $p = 0$. Consider the following familiar functional forms for $U(.)$ and $V(.)$.

$$U(c_0) = \frac{c_0^{1-\alpha}}{1-\alpha} \tag{12}$$

and

$$V(z) = \frac{z^{1-\varepsilon}}{1-\varepsilon} \tag{13}$$

Note that the reciprocal of α is the elasticity of intertemporal substitution and ϵ is the measure of proportional risk aversion. In this case, the indirect utility function reduces to[9]:

$$J(G, y_0) = \frac{y_0^{1-\alpha}}{1-\alpha}\left[1 + \widehat{R}^{\frac{1-\alpha}{\alpha}}\right]^{\frac{1}{\alpha}} \tag{14}$$

where

$$\widehat{R} = \left[\int R^{1-\varepsilon} dG(R)\right]^{\frac{1}{1-\varepsilon}} \tag{15}$$

Note that $\widehat{R}$ is the certainty equivalent rate of return.

Now $J(G, y_0)$ may be convex or concave in G depending on the values of α and ε. It is straightforward to verify that for $\alpha = \varepsilon$, it becomes convex in G. This is the special case of expected utility, when the relative risk aversion parameter ε exactly equals the inverse of the elasticity of substitution.

Proposition 3: If $logR$ is distributed normally: $N(\mu, \sigma^2)$ then

$J(G)$ is convex.

Proof: See Appendix.

[9]See Basu and Ghosh (1994) for details of the derivation.

(ii) CES-CARA case (Weil,1993):

Assume that R has a degenerate distribution and p is random. Also, assume that $U(.)$ has the same CES form as (11) but $V(.)$ has the following CARA form:

$$V(c_1) = -\lambda e^{-\lambda c_1} \tag{16}$$

In this case, the indirect utility function reduces to:

$$J(G, y_0) = \frac{\left[\frac{R^{\frac{(\alpha-1)}{\alpha}} y_0 + QR^{-\frac{1}{\alpha}}}{1+R^{\frac{\alpha-1}{\alpha}}}\right]^{1-\alpha} + \left[\frac{R.y_0 + Q\left[1-R^{\frac{\alpha-1}{\alpha}}\right]}{1+R^{\frac{\alpha-1}{\alpha}}}\right]^{1-\alpha}}{1-\alpha} \tag{17}$$

where

$$Q = -\frac{1}{\lambda} \ln \int [e^{-\lambda p} dG(p)] \tag{18}$$

Q is the certainty equivalent income and it is linear in probability .

Proposition 4: If $p \sim N(\mu_p, \sigma_p^2)$, then J is concave in G.

Proof. See Appendix. ■

These two examples amply illustrate that in a NEUM framework, the indirect utility function can be either concave, or convex in moments. Thus, convexity in moments is a necessary condition for EUM but not sufficient.

2.3 Application Methodology

Since one of the purposes of this paper is to provide a framework for *empirical* applications, it is both convenient and natural to work with an indirect utility function defined over moments. Thus, given that the moments uniquely characterize the distribution G, the solution to problem (1) can be described by the corresponding indirect utility function, H, defined as:

$$Max_{c_0}\ \{\mathcal{U}[G_c] : c_1 = (\overline{p} + e_p) + (y_0 - c_0)(\overline{R} + e_r)\} \equiv H(y_0, \overline{p}, \overline{R}, m_{-1}) \tag{19}$$

where m_{-1} is the vector of moments, other than the first and $\overline{p} = E(p)$, $\overline{R} = E(R)$.

It is convenient to write the problem, alternatively, as:

$$Max_s\ \{\mathcal{U}[G_s] : c_1 = (\overline{p} + e_p) + s(\overline{R} + e_r)\} \equiv \pi(\overline{p}, \overline{R}, m_{-1}) \tag{20}$$

where $s = y_0 - c_0$ and G_s[10] is the (cumulative) distribution of (s, c_1), which can be obtained from the distribution G_c (of (c_0, c_1)) and m_{-1} is the vector of higher order moments of p and R.

[10]Note that although we have a joint distribution, $G_s(s, c_1)$, s is a degenerate random variable.

The savings function can now be derived directly from π. Applying the envelope theorem and taking the utility function to be once differentiable,[11] we get:

$$\frac{\partial \pi}{\partial \overline{p}} = \mathcal{U}'(G_s)\frac{\partial G_s}{\partial \overline{c}_1}\frac{\partial \overline{c}_1}{\partial \overline{p}} = \mathcal{U}'(G_s)\frac{\partial G_s}{\partial \overline{c}_1} \tag{21}$$

$$\frac{\partial \pi}{\partial \overline{R}} = \mathcal{U}'(G_s)\frac{\partial G_s}{\partial \overline{c}_1}\frac{\partial \overline{c}_1}{\partial \overline{R}} = \mathcal{U}'(G_s)\frac{\partial G_s}{\partial \overline{c}_1}s \tag{22}$$

where $\overline{c}_1 = E(c_1) = \overline{p} + (y_0 - c_0)\overline{R}$.

The savings function is, therefore, given by (Roy's identity):

$$s = \frac{\partial \pi}{\partial \overline{R}} / \frac{\partial \pi}{\partial \overline{p}} \tag{23}$$

To implement the model empirically, we first have to specify a functional form for the indirect utility function, π. Given this functional form, if the moments of G are known, we could simply estimate the system of equations (23). Since the moments of the distribution G are, however, not known, they first have to be estimated. Using estimates of the moments, the savings function can, then, be estimated and used to examine the effects of changes in the moments of the distribution on savings. Furthermore, by testing for convexity in moments, we can see whether behavior is consistent with expected utility maximization.

It is important to notice that the above framework is very general in several ways. First, it is not restricted to a particular type of non-expected utility model. Second, it does not require time separability/additivity of consumption. Finally, its application provides an explicit savings function for any underlying preferences.

3 Empirical Application:

3.1 Econometric Specification

Having discussed the theoretical framework, we now provide an empirical example. The example applies our two period model to estimate a cross-country savings function using a sample of 69 countries. We chose a sample of countries for which the longest time series are available for per capita real gross domestic savings, per capita real GDP and real interest rates.[12] The real interest rate series is calculated by subtracting the CPI rate of inflation from the deposit rates for each country. The savings and GDP data were already

[11] To be able to apply the model empirically, we have to assume differentiability. For example, we can assume that the optimal solution for c_0 is, in addition to being upper semi-continuous, also unique. The indirect utility functions are then differentiable.

[12] Since the empirical model aims to estimate a houshold savings function, deposit rates seem to be the apppriate measure of interest rates. Since for the United States, no deposit rates series were available, we used prime rate instead.

adjusted for real exchange rates and are, therefore, internationally comparable. All the data came from the World Development Indicators over a sample period of 1980-99.

In order to estimate a cross country savings function which is consistent with our application methodology, we first computed the mean per capita savings for each country over the relevant sample period. In the next step, we computed the first three central moments of per capita GDP and the real interest rates, as well as the covariance between these two series over the same sample period.[13] Since it seems rather unlikely that fourth, or higher order moments will play a role in the decision,[14] we restrict ourselves to moments of an order $r \leq 3$.

Assuming that the savings, income and interest rate series for all countries are drawn from the same stationary distribution,[15] we proceed with the estimation of the cross country savings function as follows. We assume that the function $\pi(\overline{p}, \overline{R}, m_{-1})$ can be represented (approximated) by a quadratic functional form. Let the first three central moments and the covariance of p and R be given by: m_{ij}, $i = p, R$, $j = 1...3$, and σ respectively. Let the third cross moments[16] be given by m_{p^2R}, m_{pR^2}. The quadratic indirect utility function is then given by:

$$\begin{aligned} \pi \;=\;& a_0 + \sum_{j=1}^{3} a_j m_{pj} + \sum_{j=1}^{3} b_j m_{Rj} + b_\sigma \sigma + a_{21} m_{p^2R} + a_{12} m_{pR^2} \\ & + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{jk} m_{pj} m_{pk} + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} b_{jk} m_{Rj} m_{Rk} + \sum_{j=1}^{3}\sum_{k=1}^{3} g_{jk} m_{pj} m_{Rk} \\ & + \sum_{j=1}^{3}\sum_{i=p,R} c_{ij} m_{ij}\sigma + \sum_{j=1}^{3}\sum_{i=p,R} n_{2ij} m_{p^2R} m_{ij} + \sum_{j=1}^{3}\sum_{i=p,R} n_{1ij} m_{pR^2} m_{ij} \end{aligned} \tag{24}$$

where m^i_{kj}, $i = p, R$, $k, j = 1...3$, and σ respectively.

Applying Roy's identity we get the savings function as:

$$s = \frac{b_1 + \sum_{j=1}^{3} b_{1j} m_{Rj} + \sum_{j=1}^{3} g_{1j} m_{pj} + c_{r1}\sigma + h_{21} m_{p^2R} + h_{12} m_{pR^2}}{a_1 + \sum_{j=1}^{3} a_{1j} m_{pj} + \sum_{j=1}^{3} g_{1j} m_{Rj} + c_{p1}\sigma + k_{21} m_{p^2R} + k_{12} m_{pR^2}} \tag{25}$$

Since the savings equation is homogeneous of degree zero in the parameters we use the normalization $a_1 = 1$.

[13] A possible alternative would be to use a GARCH type model to estimate the moments of the distribution for each country (given the country's time series data). It is also possible to follow Appelbaum and Ullah (1997) and obtain non-parametric estimates of these moments. However, because of the two period nature of the problem, a time varying moment is not found appropriate in the present context.

[14] In fact, no fourth order of the utility function plays a role in the theory of choice under uncertainty. For examples of models with skewness, see Roll and Ross (1983), and Singleton and Wingender (1986). For an empirical model with the first four moments see Appelbaum and Ullah (1997).

[15] This is equivalent to assuming that all countries in the sample share a long run relationship between savings, income and interest rates. This long run relationship may not be necessarily linear as commonly assume in the cointegration literature. The flexible functional form allows the savings coefficients to vary across countries.

[16] Covariance between p^2 and R and between R^2 and p.

It should be noted that continuity completely characterizes the functional $J(G)$.[17] Thus, any functional with this property is the indirect utility functional for some preferences. But, since (given boundedness of p, R) the moments completely characterizes G, it follows that continuity also completely characterize $\pi(m)$, that is: any function with this property is the indirect utility function for some preferences.

For empirical implementation the model has to be imbedded within a stochastic framework. To do this, we assume that equation (25) is stochastic due to "errors in optimization". We define the "optimization errors", at observation i, as $v(i)$. We assume that $v(i)$ is identically and independently distributed with mean zero and non-singular covariance matrix.

3.2 Empirical Results

Given the wide range of countries in our sample (some very rich, others very poor), we introduce a dummy variable (whose corresponding parameter is d) to capture the country's degree of "development". We divide the sample into four quartiles, based on GDP per capita. Each country is then assigned its quartile's dummy variable.

Using the estimates of the first three moments (including the covariances), we estimate equation (25) with the additional dummy variables, using maximum likelihood[18]. The parameter estimates are given in Table 1.

Convexity in moments is a necessary (but not a sufficient) condition for expected utility maximization. To check for convexity in moments we need to obtain the parameters corresponding to the first three moments. Unfortunately, due to the differentiation, all the parameters in (24) which do not involve, m_{R1}, or m_{p1}, do not appear in the estimated saving equation (25). Consequently, we cannot check for convexity in *all* moments. It is, however, very easy to check for convexity in the first moments of the distribution. The requirement that b_{11} and a_{11} are positive is a *necessary* condition for convexity in *all* moments. As can be seen in Table 1, both b_{11} and a_{11} are significantly negative. Thus, for this example, we must reject expected utility: the model is not consistent with expected utility maximization.

3.3 Elasticities:

To examine the effects of the exogenous variables on savings, we use the parameter estimates to calculate the corresponding elasticities, $\theta_h \equiv \frac{\partial \log(s)}{\partial \log(h)}$, where $h = m_{pj}, m_{Rj}, \sigma,\ j = 1, 2, 3$. The calculated elasticities, evaluated at the quartile average are given in Table 2.

First, we check to see whether savings are affected by the higher moments, $m_{pj}, m_{Rj},\ j = 2, 3$, covariance and the third cross moments. Savings are unaffected by the higher mo-

[17] See Machina (1984) p. 208, Theorems 2 and 3 for the expected utility case.

.

[18] Taking into account hetroskedasticity.

ments if and only if the corresponding elasticities satisfy: $\theta_h = 0$ at all data points, that is, for all $h = m_{pj}, m_{Rj}, \sigma,\ m_{pR^2},\ m_{p^2R},\ j = 2,3$. The conditions for this to be satisfied globally (at all sample points) are:

$$\begin{aligned} b_{1j} &= a_{1j} = g_{1j} = c_{r1} = c_{p1} \\ &= h_{21} = h_{12} = k_{21} = k_{12} = 0 \\ j &= 2,3 \end{aligned} \tag{26}$$

The conditions for this to be satisfied locally (at a given sample point, i.e., for a given country), for the effects of the moments of R are given by:

$$\theta_{rj} \equiv \frac{\partial \log(s)}{\partial \log(m_{rj})} = \frac{\partial s}{\partial m_{rj}} \frac{m_{rj}}{s} = 0 \tag{27}$$

$$\text{which implies} \tag{28}$$

$$\frac{\partial s}{\partial m_{rj}} = 0\,,\ \ j = 2,3 \quad \text{at a given data point}$$

This can be written as:

$$\begin{aligned} [-1_1 + \sum_{i=1}^{3} a_{1i} m_{pi} + \sum_{j=1}^{3} g_{1i} m_{ri} + c_{p1}\sigma + k_{21} m_{p^2R} + k_{12} m_{pR^2}] b_{1j} &= \\ [b_1 + \sum_{i=1}^{3} b_{1i} m_{ri} + \sum_{i=1}^{3} g_{1i} m_{pi} + c_{r1}\sigma + h_{21} m_{p^2R} + h_{12} m_{pR^2}] g_{1j},\ \ j &= 2,3 \end{aligned} \tag{29}$$

We test and reject the hypothesis that savings are unaffected by the higher moments of R for at the sample average ($\chi^2 = 19.02$, $\chi^2_{(2,.001)}$ =13.81). Given that the local restrictions are rejected, it is clear that the global restrictions will be rejected as well.

The conditions that savings in a given country are (locally) unaffected by the higher moments of p are given by:

$$\theta_{pj} \equiv \frac{\partial \log(s)}{\partial \log(m_{pj})} = \frac{\partial s}{\partial m_{pj}} \frac{m_{pj}}{s} = 0 \tag{30}$$

$$\text{which implies} \tag{31}$$

$$\frac{\partial s}{\partial m_{pj}} = 0,\ \ j = 2,3 \quad \text{at a given data point}$$

This can be written as:

$$\begin{aligned} [-1_1 + \sum_{i=1}^{3} a_{1i} m_{pi} + \sum_{i=1}^{3} g_{1i} m_{ri} + c_{p1}\sigma + k_{21} m_{p^2R} + k_{12} m_{pR^2}] g_{1i} &= \\ [b_1 + \sum_{i=1}^{3} b_{1i} m_{ri} + \sum_{i=1}^{3} g_{1i} m_{pi} + c_{r1}\sigma + h_{21} m_{p^2R} + h_{12} m_{pR^2}] a_{1i},\ \ j &= 2,3 \end{aligned} \tag{32}$$

Again, we test and reject the hypothesis that savings are unaffected by the higher moments of p, at the sample average ($\chi^2 = 15.1$, $\chi^2_{(2,.001)}$ =13.81). Given that the local restrictions are rejected, it is clear that the global restrictions will be rejected as well.

Finally (and similarly), the conditions that savings in a given country are (locally) unaffected by the covariance are given by:

$$\begin{aligned} & [-1+\sum_{j=1}^{3} a_{1j} m_{pj} + \sum_{j=1}^{3} g_{1j} m_{rj} + c_{p1}\sigma + k_{21} m_{p^2R} + k_{12} m_{pR^2}] c_{r1} \\ = \; & [b_1 + \sum_{j=1}^{3} b_{1j} m_{rj} + \sum_{j=1}^{3} g_{1j} m_{pj} + c_{r1}\sigma + h_{21} m_{p^2R} + h_{12} m_{pR^2}] c_{p1} \end{aligned} \tag{33}$$

We test and reject the hypothesis that savings are unaffected by the covariance at the sample average ($\chi^2 = 12.8$, $\chi^2_{(1,.001)}$ =10.81). Given that the local restrictions are rejected, it is clear that the global restrictions will be rejected as well.

Table 2 reports some estimate elasticities, which are of theoretical interest. We compare the response of saving to interest and income risk for rich and poor countries. Poor countries are defined as the countries in bottom quartile of the world distribution of income while the rich countries are those in the top quartile range of income. We focus here on two measures of risk: (i) a symmetric risk which is captured by a mean preserving spread; (ii) non-symmetric or downside risk, which is reflected by the skewness or the third moment. Increase in riskiness of interest rate measured in either of these ways lowers savings for poor countries, while the effect is the opposite for rich countries. This heterogeneity in the response of savings to interest risk among rich and poor countries cannot be rationalized by a standard constant elasticity type of utility functions as used by Selden (1978) and others. The reason is that the measures of elasticity of substitution and risk aversion are assumed to be independent of the level of income. However, it may be possible to explain this phenomenon with a more general and flexible class of preferences, which allows the elasticity of substitution to depend on income and wealth and interact with risk aversion.[19]

The effect of income risk on saving shows an opposite pattern in the data. Poor countries save more than rich countries in response to a higher income risk (measured either by a mean preserving spread of income or a lower skewness). This kind of behavior is consistent with a decreasing absolute risk aversion (DARA) class of utility function.[20] It is important to observe that all these comparative statics results cannot be necessarily

[19]Response of savings to interest rate risk seems to contradict the existing finding that poor countries may have a lower elasticity of substitution than rich countries as found by Atkeson and Ogaki (2001). However, it may be possible to exlain our emoirical resukts n terrms of more general NEUM which allows for nonlinear interaction between risk aversion and intertemporal substitution.

[20]Alm (1988), and Kim et al. (1996) derive the effects of income risk on savings when households have DARA type utility function. Menezes et al. (1980) show a DARA preference means aversion to downside risk of income, which means aversion to lower skewness of income in our context.

understood in terms of a unified utility function because of our approach involving a flexible functional form, which is indeed a strength (but, ironically, possibly also a weakness) of our methodology.

4 Conclusion

In this paper, we provide an empirical framework for the analysis of general non-expected utility behavior in the context of savings under uncertainty. We extend the application of existing models of savings behavior under non-expected utility to be applicable to general underlying preferences. Despite this generalization, our approach maintains tractability of the savings function by applying principles of duality. Using a flexible functional form, we test a key necessary condition for the validity of expected utility hypothesis, using cross-country savings data. The expected utility hypothesis is rejected by our data. By using a flexible functional form for the indirect utility function, we allow for heterogeneity of savings response to changes in income and interest rates across countries. The estimated savings elasticities show significant variation across rich and poor countries, a finding that raises new theoretical questions. A useful extension of this paper would be to apply our approach within a multiperiod framework[21].

5 Appendix:

Proof of Proposition 1:

Consider two distributions G_1 and G_2 and let $G_\lambda \equiv \lambda G_1 + (1-\lambda)G_2$ where and $0 \leq \lambda \leq 1$. Let the corresponding solutions to problem (5) be given by: c^1, c^2, and c^λ respectively. The "linearity in probabilities" implies that: $J(\lambda G_1 + (1-\lambda)G_2) \equiv \int_{R,p\in\mathcal{A}} U(c^\lambda) d(\lambda G_1 + (1-\lambda)G_2)$

$= \lambda \int_{R,p\in\mathcal{A}} U(c^\lambda) dG_1 + (1-\lambda) \int_{R,p\in\mathcal{A}} U(c^\lambda) dG_2$

$\leq \lambda \int_{R,p\in\mathcal{A}} U(c^1) dG_1 + (1-\lambda) \int_{R,p\in\mathcal{A}} U(c^2) dG_2$

$= \lambda\, J(G_1, y_0) + (1-\lambda)\, J(G_2, y_0)$, i.e., $J(G)$ is convex (consequently, it also follows from the result in Rockafellar (1970), p. 82, that it is continuous in G). See Kreps and Porteus (1979).

Proof of Proposition 2: Given the distributions G_1, G_2 and G_λ, let the corresponding distributions of c be given by G_c^1, G_c^2, and G_c^λ. Let the corresponding solutions to problem 5 be given by: c^1, c^2 and c^λ respectively. In other words, $J(G_\lambda, y_0) \equiv \mathcal{U}[G_c^\lambda(c^\lambda)]$, $J(G_1, y_0) \equiv \mathcal{U}[G_c^1(c^1)]$ and $J(G_2, y_0) \equiv \mathcal{U}[G_c^2(c^2)]$. From the definitions of the maximizers it follows that: $\mathcal{U}[G_c^1(c^\lambda)] \leq \mathcal{U}[G_c^1(c^1))] \equiv J[G_1, y_0]$ and $\mathcal{U}[G_c^2(c^\lambda)] \leq \mathcal{U}[G_c^2(c^2))] \equiv J[G_2, y_0]$. But now, since we do not have "linearity in probabilities", we cannot "break up" the indirect utility into two terms, as was done above: $J[G_\lambda, y_0] \neq \lambda\, J(G_1, y_0) + (1-\lambda)\, J(G_2, y_0)$. In general, it is possible to have $J[G_\lambda, y_0]$ smaller, greater

[21] This is currently studied by the authors.

or equal to$\lambda\, J(G_1, y_0) + (1-\lambda)\, J(G_2, y_0)$, which means that the functional J may, or may not be convex in G.

Whether J is convex depends on the specific non-expected utility model that is being used. Consider, for example, the following non-expected utility models:

1. Weighted Utility (Karmarkar (1978) and Chew (1983))[22]
2. Rank Dependent Expected Utility (Quiggin (1982))[23]
3. Quadratic (Machina 1982)[24]
4. Regret/Rejoice (no transitivity), e.g., Fishburn (1983)[25].

In all of the cases above we no longer have linearity in probabilities. Consequently, the functional $J(G)$ is not *necessarily* convex. Whether it is convex or concave depends on the specific non-expected utility model that is chosen, on its parameters and on the properties of the "probability transformation" function.

Proof of Theorem 1: Consider two distributions, G_1 and G_2, whose characteristic functions are Ψ_1 and Ψ_1. For any $0 \leq \lambda \leq 1$, define $G^\lambda \equiv \lambda G_1 + (1-\lambda) G_2$ and let Ψ_λ be the characteristic function corresponding to G^λ. It is known (see Feller (1966), Lemma on p. 477) that for any $0 \leq \lambda \leq 1$, $\Psi_\lambda = \lambda\Psi_1 + (1-\lambda)\Psi_2$. Now, as is well known (see, for example, Uniqueness and Inversion Theorems in Wilks (1964) pp. 116-118 and Feller (1966), pp. 480-482), distribution and characteristic functions uniquely determine each other. It, therefore, follows that the distribution functions corresponding to Ψ_λ and $\lambda\Psi_1 + (1-\lambda)\Psi_2$ are the same. Let us define the distribution that corresponds to $\lambda\Psi_1 + (1-\lambda)\Psi_2$ as $G_{\lambda\Psi_1+(1-\lambda)\Psi_2}$. Hence, since the distribution corresponding to Ψ_λ is $\lambda G_1 + (1-\lambda) G_2$ and the distribution corresponding to $\lambda\Psi_1 + (1-\lambda)\Psi_2$ is, by definition, $G_{\lambda\Psi_1+(1-\lambda)\Psi_2}$, it follows that $\lambda G_1 + (1-\lambda) G_2 = G_{\lambda\Psi_1+(1-\lambda)\Psi_2}$. Or alternatively, since $G_1 \equiv G_{\Psi_1}$ and $G_2 \equiv G_{\Psi_2}$ (the distribution function that corresponds to the characteristic function whose distribution function is G_i, is G_i itself) we have $\lambda G_{\Psi_1} + (1-\lambda) G_{\Psi_2} = G_{\lambda\Psi_1+(1-\lambda)\Psi_2}$. In other words, the distribution function is linear in Ψ.

Now, since $R, p \in \mathcal{A}$ is concentrated in a finite interval, the characteristic function Ψ is analytic (see Laha and Rohatgi (1979) Theorems 4.2.2. and 4.2.3. pp. 253-254, Wilks (1964) p. 126) and can be written as a sum (which converges) whose coefficients are the moments: $\Psi_G(t) = \sum_r (\mathrm{i}t)^r m_r / r!$. Since this series is linear in the coefficients (the moments m_r), it follows that the characteristic function is linear in moments. Thus, since (i) J is convex in G, (ii) the distribution function is linear in Ψ, and (iii) the characteristic function, is linear in the moments, it follows that H is convex in the moments.[26]

---

[22]For example, the discreet version (assuming that c can obtain the values $c_1, ...c_n \equiv c'$ with probabilities $q_1...q_1$) of Karmarkar (1978) and Chew (1983) are given by $\mathcal{U}[c', q] = \frac{\sum_i v(c_i)\ f(q_i)}{\sum_i f(q_i)}$ and $\mathcal{U}[c', q] = \frac{\sum_i v(c_i) q_i\ \ f(c_i)}{\sum_i q_i\ \ f(c_i)}$ respectively.

[23]For example, the discreet version of Quiggin (1982) is given by: $\mathcal{U}[c', q] = \sum_i v(c_i)\{f(\sum_{j=1}^{i} q_j) - f(\sum_{j=1}^{i-1} q_j)]$, where $f : [0,1] \to [0,1]$ is increasing and continuous probability weighting function.

[24]For example, the discreet version of Machina (1982) is given by: $\mathcal{U}[c', q] = \sum_i q_i v(c_i) + \{\sum_i q_i v^{'}(c_i)\}^2$.

[25]For example, the discreet version of Fishburn (1982) is given by: $\mathcal{U}[c', q] = \sum_i v(c_i; q)\ \ f(q_i)$.

[26]In addition, it also follows from the Continuity Theorem (see Feller (1966), pp 481, Theorem 2) that

$\mathcal{Q.E.D}$.

Proof of Theorem 2: The first part of the proof is the same as above: the distribution function is linear in Ψ and the characteristic function is linear in the moments. But now, since J is not necessarily convex in G, H may be either convex or concave in the moments. For example, consider the Rank Dependent Expected Utility case. Since the mean and probabilities (q) are linearly related, we know that when J^0 is concave in q, then H will be concave in the mean, whereas when J^0 is convex q, then H will be convex in the mean. A similar example can be provided for the other non-expected utility cases. $\mathcal{Q.E.D}$.

Proof of Proposition 3: Using the lognormality property 15 can be written as:

$$\widehat{R} = \exp\left(\mu_r + 0.5(1-\varepsilon)\sigma^2\right) \tag{34}$$

Plugging 34 into 14, one obtains:

$$H(m, y_0) = \frac{y_0^{1-\alpha}}{1-\alpha}\left[1 + \exp\left(\mu_r \frac{1-\alpha}{\alpha} + 0.5\frac{(1-\varepsilon)(1-\alpha)\sigma_r^2}{\alpha}\right)\right]^{\frac{1}{\alpha}} \tag{35}$$

It is now straightforward to verify that $H(m, y_0)$ is convex in μ_r and σ_r^2. Next, note that the kth moment of R is given by:

$$m_k = \exp(k\mu_r + 0.5k^2\sigma_r^2) \tag{36}$$

which means all the moments are convex in μ_r and σ_r^2. Following the same steps as in the proof of Theorem it therefore, follows that $H(m, y_0)$ is convex in m. $\mathcal{Q.E.D}$..

Proof of Proposition 4:

Note that(18)reduces to:

$$Q = \mu_p - 0.5\lambda\sigma_p^2 \tag{37}$$

Substituting 18 in 17 one obtains

$$H(m, y_0) = \frac{\left[\frac{R^{\frac{(\alpha-1)}{\alpha}}.y_0 + \left(\mu_p - 0.5\lambda\sigma_p^2\right).R^{-\frac{1}{\alpha}}}{1+R^{\frac{\alpha-1}{\alpha}}}\right]^{1-\alpha} + \left[\frac{R.y_0 + \left(\mu_p - 0.5\lambda\sigma_p^2\right)\left[1-R^{\frac{\alpha-1}{\alpha}}\right]}{1+R^{\frac{\alpha-1}{\alpha}}}\right]^{1-\alpha}}{1-\alpha} \tag{38}$$

Note that H is concave in in mean and variances of p. Since the normal distribution belongs to a two-parameter family, mean and variance characterize all the moments. Thus H is concave in m. Since there is a one-to-one correspondence between the moments and the distribution function via the characteristic function, J is concave in G when G is normal. $\mathcal{Q.E.D}$..

H is continuous in m.

Table 1: Parameter Estimates

Parameter	Estimate	t-statistic
b1	-.354281	-3.47462
b11	-.036477	-2.57270
b12	.107101	2.58383
b13	.649288E-02	.629253
g11	.080940	4.40647
g12	-.189609	-1.24735
g13	-.029734	-.356578
h11	.184220E-03	.163832E-02
h21	.180815E-02	1.28945
h12	-.142277E-02	-1.58785
a11	-.144772	-4.74842
a12	-.323142	-.864344
a13	-.121042	-.509222
g21	-.326236	-7.08066
g31	-.023795	-1.17974
k11	-.821454E-02	-2.13453
k21	.446119E-02	1.71475
k12	.489602E-02	4.10647
d	1.15278	6.36491
Adjusted R-squared	.989090	
Durbin-Watson	1.27089	

Table 2: Estimated Elasticities

Countries	Low Income (Bottom 25%)	High Income (Top 25%)
$E(R)$	-0.204	2.234
$Var(R)$	-0.035	1.96263
$Skew(R)$	0.018	-0.078
$E(p)$	0.654	3.064
$Var(p)$	0.037	-0.031
$SKEW(p)$	-0.028	0.0002

6 References

Alm, J. (1988), "Uncertain Tax Policies, Individual Behavior, and Welfare," American Economic Review, 78, 237-245.

Appelbaum, E. and A. Ullah, (1997), "Estimation of Moments and Production Decisions under Uncertainty", *Review of Economics and Statistics.*

Appelbaum, E., (1998), "An Econometric Test of Expected Utility Theory", paper presented at the Econometric Society meetings in Berlin.

Atkeson, A and M. Ogaki, “Wealth-Varying Intertemporal Elasticities of Substitution: Evidence from Panel Data and Aggregate Data ” *Journal of Monetary Economics*, 38, 507-534.

Basu, P and S. Ghosh., (1993), ”Optimal Savings Under General Changes in Uncertainty: A Non-Expecetd Utility Maximizing Approach”, *Southern Economic Journal*, 60, 421-439.

Berg, C., (1963), *Topological Spaces, tr. by Patterson*, New York, MacMillan.

Camerer, C.F., (1989), “An Experimental Test of Several Generalized Utility Theories”, *Journal of Risk and Uncertainty*, 2, 61-104.

Chew, S.H., (1983), “A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox”, *Econometrica, 51, 1065-92.*

Epstein L.G., (1992), “Behavior under Risk: Recent Developments in Theory and Applications”, in *Advances in Economic Theory, sixth world congress*, Vol, 2, J-J Laffont, ed.

Epstein L.G. and S. E. Zin, (1991), “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and asset Returns: An Empirical Analysis”, *Journal of Political Economy*, Vol. 99, 263-286.

Feller, William, (1966), *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York.

Fishburn, P.C., (1983), “Transitive Measurable Utility”, *Journal of Economic Theory*, 31, 293-317.

Geuven. M.F., (2001), ”Mismeasurement of Elasticity of Intertemporal Substitution: The Role of Limited Stock Market Participation,” working paper, Carnegie Mellon University.

Hall, R.E. (1988), ”Intertemporal Substitution in Consumption,” *Jounral of Political Economy*, 96,2, 339-357.

Karmarkar, U.S., (1978), ”Subjectively Weighted Utility A Descriptive Extension of the Expected Utility Model”, *Organizational Behavior and Human Performance,* 21, 61-72.

Kahneman, D. and A. Tversky, (1979), Prospect Theory: An Analysis of Decision under Risk”, *Econometrica*, 47, 263-91.

Kendall, M.G., (1969), *The Advanced Theory of Statistics,* Vol 1, Hafner publishing Company, New York.

Kim,I., Snow, A., and Warren Jr, R.S.,”Tax Rate Uncertainty,Factor Supplies, and Welfare,*Economic Inquiry*, 33, 159-69.

Kreps, D.M., (1990), *A Course in Microeconomic Theory.* Princeton University Press.

Kreps, D.M and E.L. Porteus, (1979), ”Temporal von Neumann-Morgenstern and Induced Preferences”, *Journal of Economic Theory*, 20, 81-109.

Laha, R.G. and V.K. Rohatgi, (1979), *Probability Theory*, John Wiley & Sons.

Machina, M., (1982), "Expected Utility Analysis without the Independence Axiom", *Econometrica*, 50, pp. 277-323.

Machina, M., (1984), "Temporal Risk and Induced Preferences", *Journal of Economic Theory*, 33, pp. 199-231.

Machina, M., (1987), "Choice under Uncertainty: Problems Solved and Unsolved", *Journal of Economic Perspectives*, 1, pp. 121-54.

Mass-Collel, A., Whinston, M.D. and J.R. Green, (1995), *Microeconomic Theory*, Oxford University Press.

Mehra R. and E.C. Prescott (1985), "The Equity Premium: A Puzzle," Journal of Monetary Economics 15: 145-161.

Menzes, C.F.,C.Geiss and J. Tressler, "Increasing Downside Risk," American Economic Review, December 1980, 921-32.

Poterba, J and A.A. Samwick (1995), "Stock Ownership Patterns, Stock market Fluctuations, and Consumption," Brookings Papers on Economic Activity, pp. 295-373.

Quiggin, J,. (1982), "A Theory of Anticipated Utility", *Journal of Economic Behavior and Organization*, 3, 323-43.

Rockafellar, R.T., (1970), Convex Analysis, Princeton University Press, Princeton, N.J.

Selden, L. (1979), "An OCE Analysis of the Effect of Uncertainty on Saving under Risk Independence," *Review of Economic Studies*, 46, 73–82.

Tversky, A. and D. Kahneman, (1986), Rational Choice and the Framing of Decisions", *Journal of Business*, 59, 251-78.

Weil, P., (1989), "The Equity Premium Puzzle and the Risk-free Rate Puzzle," *Journal of Monetary Economics,* 24, 401-421.

Weil, P., (1993), "Precautionary Savings and the Permanent Income Hypothesis," *Review of Economic Studies*, 60, 367-383.

Wilks, S.S, (1964), *Mathematical Statistics*, John Wiley & Sons, New York.