Zheng's Optimal Mechanism with Resale and the Second-Price Auction.

by Bernard Lebrun[1]
Department of Economics, York University, 4700 Keele Street, Toronto, ON Canada M3J 1P3; blebrun@econ.yorku.ca

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Abstract

We show that Zheng (2002)'s optimal mechanism in the presence of resale can be interpreted as an equilibrium of an ascending-price auction and, in the two-bidder case, as an equilibrium with a no-regret property of the English and second-price auctions. We also show that it can be extended beyond Zheng (2002)'s original assumptions.

Keywords: Zheng's mechanism, optimality, resale, second-price auction, independent private values.

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1. Introduction

In the independent private value model with possible resale between bidders, Zheng (2002) constructs a selling mechanism that, under some assumptions, implements Myerson (1981)'s allocation. The mechanism is thus optimal among all selling mechanisms, even those that can prevent resale. This result is important since, in many cases, resale is too costly or impossible to prevent.

We first interpret, in Section 2, Zheng 's mechanism as an equilibrium of a non-standard ascending-price auction with sequential entry. In Sections 3 and 4, we then show that, with 2 bidders, it is equivalent to an equilibrium, with a no-regret property, of the standard English and second-price auctions.

Furthermore, the English and second-price auctions implement the optimal allocation under more general assumptions than Zheng (2002)'s. In fact, from the three assumptions for the two-bidder case in Zheng (2002), we only keep unchanged Assumption 3–the Resale Monotonicity Assumption. We replace Assumption 1, which requires that the hazard rates be strictly increasing, by the standard assumption that the virtual-value functions be strictly increasing (the "regular case" in Myerson, 1981). We drop Assumption 2–the Uniform Bias Assumption–, according to which the valuation supports are nested and the hazard rates are ranked. Consequently, we allow valuation distributions with overlapping supports and without relation of stochastic dominance between them.

We show in Section 5 how Zheng's mechanism itself can be amended to deal with such less restricted distributions.

Our results follow from a recent contribution on standard auctions with resale (Lebrun, 2006), which, through a link with the common-value model, characterizes an infinity of equilibria of the second-price auction with resale.

Mylovanov and Tröger (2006) investigate the generality of Zheng (2002)'s assumptions, especially in the n-bidder case. Calzolari and Pavan (2005) obtain optimal mechanisms with resale for a two-bidder model where the values are distributed over two-point supports and where the bargaining powers at resale depend on the bidders' identities. They then implement their optimal mechanisms through, depending on the values of their parameters, first-price auctions with limited sets of acceptable bids (and specific tie-breaking rules) or second-price auctions with personalized reserve prices.

2. Zheng's Mechanism as an Ascending-Price Auction.

In Zheng (2002), the bidders' hazard rates are ranked: a lower index bidder has a higher hazard-rate function. Starting with bidder 1, Zheng 's mechanism examines the bidders sequentially according to increasing index order and awards the item to the first bidder i who passes the following test: the transformation, through the pre-specified function β_{ij}, of the value bidder i submits is not smaller than the value bidder j submits, for all $j > i$.

A bidder is called the "leader" when it is his turn to be examined. When the current leader passes the test, the price he pays for the item is, if he is the highest index bidder, his personalized reserve price or, if he is not, a pre-announced and specific nondecreasing transformation of his bid. When he fails the test, the mechanism updates the reserve prices of all remaining possible winners and examines the next leader. At the equilibrium of this mechanism, the bidders truthfully reveal their values.

This mechanism can be naturally interpreted as an equilibrium of an acsending-price auction that allows only two bidders at a time and starts, at the initial stage, with bidders 1 and 2. The high bidder may stay on for the next stage, where the new bidder allowed in will be the bidder whose index comes next after the maximum of the current-stage bidders' indices[2].

[2]Under Zheng (2002)'s "Transitivity Assumption," we do not need, when there is a change of leader to compare the new leader's transformed value with bidders' values that have already been compared with the previous leader. The same assumption implies here that the high-bidder will bid higher in the next stage than in the current stage. From the

If there is no such new bidder, the high bidder wins the auction and pays a price calculated from his bid and his reserve price. The leader is, in this interpretation, the current high bidder.

Every time the stage high bidder changes, the auctioneer informs the potential participants to the future stages of their reserve prices, which he has updated using the bids submitted by the losers in the previous stages. Otherwise, the bidders' actions are kept secret. The rules governing the updating of the reserve prices and the winner's payment can be chosen to obtain an equilibrium that mimics the honest equilibrium of Zheng's mechanism, that is, where bidder i bids according to the function β_{ij}, $j > i$, if he is the previous-stage high bidder and is matched with bidder j and bids his value if he is the new bidder.

Obviously, this auction with sequential entry is very different from the standard ascending-price auctions: the price may increase discontinuously at the start of a new stage, at no stage does the high bidder observe the low bidder's bid, and the bid of the last losing bidder does not always determine the final price. In the next sections, we show that, at least in the two bidder case, Zheng's mechanism is exactly equivalent to an equilibrium of the standard English and second-price auctions, where the winner observes and pays the maximum of the loser's bid and the reserve price.

3. The English and Second-Price Auctions with Resale

3.1 The Sealed-Bid Second-Price Auction with Deferred Payment

In this section, we extend some known results about the second-price auction. Consider the independent private values model with two, possibly heterogeneous, risk-neutral bidders. Bidder i's use value v_i for the item being auctioned is distributed over an interval $[c_i, d_i]$, with $0 \leq c_i \leq d_i$, according

ranking of the hazard rates, the new added bidder bids higher than the previous stage low bidder. In this auction interpretation of Zheng's mechanism, we may thus assume that the price is ascending and that exit is irrevocable.

to an absolutely continuous probability measure F_i, with a strictly positive, continuous, and bounded density function f_i over $(c_i, d_i]$, $i = 1, 2$. We use the same notations F_1 and F_2 for the cumulative distribution functions. A bidder's use value is his private information.

We allow a reserve price and personalized entrance fees. No bidder observes his opponent's participation decison. We add a post-auction stage where resale takes place between bidders if and only if the auction has resulted in the sale of the item and the auction winner proposes a resale price the loser agrees to.

We first consider the sealed-bid second-price auction where the auctioneer keeps the number of bids and their values secret, announces only the identity of the winner, if any, and delays the announcement of the auction price until after the resale stage. In the definition of a regular equilibrium below, r is the reserve price.

Definition 1:

(i) A regular bidding function β_i of bidder i is a real-valued function, nondecreasing and continuous from the right over $[c_i, d_i]$, equal to -1 over $[c_i, c_i')$, not smaller than r over $[c_i', d_i]$, constant over $[c_i', c_i'')$, and strictly increasing and continuous over $[c_i'', d_i]$, where c_i', c_i'' are such that $c_i \le c_i' \le c_i'' \le d_i$.

(ii) A regular resale-offer function γ_i of bidder i is a bounded and measurable (with respect to the σ-algebra of the Borel subsets) function defined over $[c_i, d_i] \times [r, +\infty)$ and such that $\gamma_i(v; b) \ge v$, for all (v, b) in $[c_i, d_i] \times [r, +\infty)$.

(iii) A regular strategy of bidder i is a couple $\sigma_i = (\beta_i, \gamma_i)$ where β_i is a regular bidding function and γ_i is a regular resale-offer function.

(iv) A regular equilibrium (σ_1, σ_2) is a couple of regular strategies that can be completed[3] into a perfect Bayesian equilibrium.

If bidder i with use value v_i follows the bidding function β_i, he participates

[3]By adding beliefs and by adding what responses every bidder gives to offers from the other bidder at the resale stage, as functions of past observed histories.

in the auction, pays his entrance fee, if any, and bids $\beta_i(v_i)$ when $v_i \geq c'_i$ and does not take part in the auction when $v_i < c'_i$. If he follows the resale-offer function γ_i, he offers $\gamma_i(v_i; b)$ at resale after winning the auction with the bid b.

Definition 2:

(i) For all $i = 1, 2$, bidder i's virtual-value function ω_i is defined over $(c_i, d_i]$ as follows:

$$\omega_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

If ω_i is strictly increasing, $\omega_i(c_i)$ is the value, possibly infinite, of its continuous extension at c_i.

(ii) Let ω_i be strictly increasing over $(c_i, d_i]$, for all $i = 1, 2$. Let the optimal-resale-price function ρ be the function defined over $[c_1, d_1] \times [c_2, d_2]$ such that, for all (w_1, w_2) in $[c_1, d_1] \times [c_2, d_2]$,

$$\rho(w_1, w_2) = \inf\left\{p \in (c_l, d_l] \,|\, w_k \leq p - \frac{F_l(w_l) - F_l(p)}{f_l(p)}\right\},$$

where k and l are such that $\{1, 2\} = \{l, k\}$ and $w_k \leq w_l$, that is, $l \in \arg\max_i w_i$ and $k \in \arg\min_i w_i$.

(iii) Notation: $\rho_1(v, w) = \rho_2(w, v) = \rho(v, w)$, for all (v, w) in $[c_1, d_1] \times [c_2, d_2]$.

In Definition 2 (ii), $\rho(w_1, w_2)$ is the resale price that maximizes bidder k's expected payoff when bidder k's use value is w_k and bidder l's use value is distributed according to F_l conditionally on belonging to the interval $[c_l, w_l]$. Indeed, in this case, bidder k's maximization problem is equivalent to $\max_{p \in [w_k, w_l]} (p - w_k)(F_l(w_l) - F_l(p))$, whose first-order condition is (ii). According to the notation (iii), ρ_i is the function ρ with bidder i's use value as its first argument.

In Result 1 below, $\alpha_i = \beta_i^{-1}$ is the "extended" inverse of β_i that is continuous from the right, that is, $\beta_i^{-1}(b) = \sup\{v_i \in [c_i, d_i] \,|\beta_i(v_i) \leq b\}$, for all b in $\{-1\} \cup [r, +\infty)$. The functions γ_1, γ_2 are the smallest optimal resale-offer functions. We show in Appendix 1 how to extend the proof in Lebrun (2006), which deals with the particular case of the same valuation interval, mandatory participation, and no reserve price or entrance fees, to our more general setting.

Result 1 (from Lebrun, 2006): *Let ω_1 and ω_2 be strictly increasing. Let c_i', c_i'' be in $[c_i, d_i)$, for all $i = 1, 2$, such that $c_1' = c_1'' \leq c_2' = \rho(c_1'', c_2'') \leq c_2''$. Let φ be a real-valued function strictly increasing and continuous over $[c_1'', d_1]$ such that $\rho(v_1, \varphi(v_1))$ is strictly increasing and $\varphi(c_1'') = c_2''$, $\varphi(d_1) = d_2$. Let r, e_1, and e_2 be a reserve price and entrance fees such that:*

$$r \leq c_1''; e_i \leq e_i(r), \text{ with } e_i = e_i(r) \text{ when } c_i' > c_i;$$

where

$$e_1(r) = (c_2' - r) F_2(c_2'') + (c_1'' - c_2') F_2(c_2'), e_2(r) = (c_2' - r) F_1(c_1'').$$

Let $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ be the following couple of regular strategies:

$$\begin{aligned}
\beta_1(v_1) &= -1, \text{ if } v_1 \in [c_1, c_1''), \\
&= \rho(v_1, \varphi(v_1)), \text{ if } v_1 \in [c_1'', d_1]; \\
\beta_2(v_2) &= -1, \text{ if } v_2 \in [c_2, c_2'), \\
&= r, \text{ if } v_2 \in [c_2', c_2''), \\
&= \rho\left(\varphi^{-1}(v_2), v_2\right), \text{ if } v_2 \in [c_2'', d_2];
\end{aligned}$$

$$\begin{aligned}
\gamma_1(v_1; b) &= \rho(v_1, \max(v_1, \alpha_2(b))); \\
\gamma_2(v_2; b) &= \rho(\max(\alpha_1(b), v_2), v_2);
\end{aligned}$$

where α_i is the extended inverse of β_i, $i = 1, 2$, for all v_1 in $[c_1, d_1]$, v_2 in $[c_2, d_2]$, and b in $[r, +\infty)$. Then, we have:

(i) (Equilibria) $(\beta_1, \gamma_1; \beta_2, \gamma_2)$ is a regular equilibrium of the second-price auction where payments are deferred, ties are broken in favor of bidder 1, and with reserve price r and personalized entrance fees e_1, e_2. Moreover, the following equalities hold true:

$$\begin{aligned} \beta_1(v_1) &= \beta_2(\varphi(v_1)), \\ \rho(\alpha_1(b), \alpha_2(b)) &= b, \ (1) \end{aligned}$$

for all v_1 in $(c_1'', d_1]$ and b in $(\rho(c_1'', c_2''), \rho(d_1, d_2))$.

(ii) (Sets of Optimal Bids) If v_i in $[c_i', d_i]$ is such that $\beta_i(v_i) < v_i$, then all bids in the interval $\left[\underline{\beta}(v_i), \max(v_i, d_i)\right]$ are optimal for bidder i with use value v_i, where $\underline{\beta}(v_i) = \max(\{r\} \cup \{w \in [c_1'', d_1] \, | \varphi(w) = w \leq v_i\})$.

(iii) (Final Equilibrium Allocations) Let λ^φ be the function defined over $[c_1, d_1]$ as follows:

$$\begin{aligned} \lambda^\varphi(v_1) &= c_2', \text{ if } v_1 < c_1''; \\ \lambda^\varphi(v_1) &= \rho(v_1, \varphi(v_1)), \text{ if } \varphi(v_1) \geq v_1 \geq c_1''; \\ \lambda^\varphi(v_1) &= \left(\rho\left(\varphi^{-1}(.), .\right)\right)^{-1}(v_1), \text{ if } \varphi(v_1) \leq v_1 \leq \min(\rho(d_1, d_2), d_1); \\ \lambda^\varphi(v_1) &= d_2, \text{ if } v_1 \geq \min(\rho(d_1, d_2), d_1). \end{aligned}$$

If $v_2 > \lambda^\varphi(v_1)$, the item goes eventually to bidder 2. If $v_2 < \lambda^\varphi(v_1)$ and $v_1 \geq c_1''$, the item eventually goes to bidder 1. If $v_2 < \lambda^\varphi(v_1)$ and $v_1 < c_1''$, the item stays with the auctioneer.

The intuition for Result 1 comes from a link between our model and the common-value model. Optimal resale under incomplete information at least remedies the "worst cases" of inefficiency, where the auction loser's use value is the highest one the winner thinks possible, that is, when both bidders submit the same bid. In this case, each bidder's net value for winning is equal to the resale price: by winning a bidder saves the resale price if he would be a buyer at resale and earns it if he would be a reseller. At an equilibrium, the first-order condition (1) thus follows. As in the common-value second-price auction, the multiplicity of equilibria described in Result 1 ensues from this, common to both bidders, condition.

To illustrate (ii), consider, for example, bidder 2 with use value v_2 such that he could not be the reseller, that is, such that $\alpha_1(\beta_2(v_2)) < v_2$. Let a bid b be such that:

$$\alpha_1(b) < \alpha_2(b) \text{ and } b < v_2. \quad (2)$$

When, at equilibrium, bidder 1 submits such a bid b and wins, he proposes (from (1)) b at resale, which bidder 2 accepts. Since the prices are identical, bidder 2 is indifferent between winning the auction against such a bidder and losing it and thus between submitting b and submitting slightly different bids. The set of optimal bids in (ii), to which $\beta_2(v_2)$ belongs, is the closure of the interval of such bids b's.

Checking that the final allocation is as in Result 1 (iii) is simple. For example, assume $v_1 \geq c_1''$ and $\varphi(v_1) \geq v_1$. From the definition of the bidding functions and of λ^φ, $\lambda^\varphi(v_1) \leq \varphi(v_1) = \alpha_2\beta_1(v_1)$. If $v_2 < \lambda^\varphi(v_1)$, bidder 2 loses the auction and refuses bidder 1's resale offer. If $\lambda^\varphi(v_1) < v_2 < \varphi(v_1)$, bidder 2 loses the auction and accepts bidder 1's resale offer. If $\varphi(v_1) < v_2$, bidder 2 wins the auction and no advantageous resale is possible.

3.2 The Standard Second-Price and English Auctions.

We now consider the second-price auction where, as it is standard, the auction winner learns the auction price–the maximum of the reserve price and the loser's bid–right at the conclusion of the auction. As in the previous subsection, the bidders' bids are kept secret. We need to extend our definition of a regular equilibrium to "behavioral" strategies.

Definition 3:

(i) A regular bidding strategy $G_i(.|.)$ *of bidder* i *is a regular conditional probability measure*[4] *with respect to* v_i *in* $[c_i, d_i]$*.*

(ii) A regular resale-offer function δ_i *of bidder* i *is a bounded and measurable function defined at all* (v_i, b_i, b) *in* $[c_i, d_i] \times [r, +\infty)^2$ *and such that* $\delta_i(v_i; b_i, b) \geq v_i$*, for all such* (v_i, b_i, b)*.*

(iii) A regular strategy of bidder i *is a couple* $(G_i(.|.), \delta_i)$ *where* $G_i(.|.)$ *is a regular bidding strategy and* δ_i *is a regular resale-offer function.*

(iv) Bidder i*'s beliefs are regular if they are represented by a regular conditional probability measure* $F_j(.|.,.)$ *with respect to* (b_i, b) *in* $[r, +\infty)^2$ *such that* $b \leq b_i$*.*

(v) A regular equilibrium is a couple of regular strategies and a couple of regular beliefs $(G_1(.|.), \delta_1, F_2(.|.,.); G_2(.|.), \delta_2, F_1(.|.,.))$ *that can be completed into a perfect Bayesian equilibrium.*

The measure $F_j(.|b_i, b)$ represents the revised beliefs bidder i holds about bidder j's use value after winning the auction with the bid b_i and learning his payment b. If bidder i with use value v_i follows $(G_i(.|.), \delta_i)$, he chooses his bid according to $G_i(.|v_i)$ and, if he has won the auction with the bid b_i and has to pay b, he offers $\delta_i(v_i; b_i, b)$ at resale. Result 2 below is an extension to the case with reserve price and entrance fees of a result in Lebrun (2006).

Result 2 (from Lebrun, 2006): *Let* ω_1 *and* ω_2 *be strictly increasing. Let* $\mathcal{E}$ *be a regular equilibrium as in Result 1 of the second-price auction with deferred payment. Then, there exists a regular equilibrium* $\mathcal{E}'$ *of the standard second-price auction with the same reserve price, entrance fees, and tie-breaking rule, such that:*

(i) (Equivalent Outcomes) The bid marginal distributions, the interim total expected payoffs, and the final allocation are the same in as $\mathcal{E}$*. Conditionally on the use value of the auction winner, resale takes place with the same probability as in* $\mathcal{E}$ *and, when this probability is different from zero,*

[4] Also called "stochastic kernel" and "transition probability distributions."

at the same price;

(iii) (No-Regret Property) In $\mathcal{E}'$, all bids in the support of $G_i(.|v_i)$ are optimal bids in $[r, +\infty)$ for bidder i even after learning his payment in case of winning, for all v_i in $[c'_i, d_i]$ and all $i = 1, 2$[5]*.*

If bidder 1's entrance fee e_1 is equal to the maximum entrance fee $e_1(r)$, then (i) and (ii) also hold true for any tie-breaking rule.

Let $\mathcal{E} = (\beta_1, \gamma_1; \beta_2, \gamma_2)$ be a regular equilibrium as in Result 1. The proof of Result 2, which we outline in Appendix 2, consists in constructing an equilibrium $\mathcal{E}' = (G_1(.|.), \delta_1, F_2(.|., .); G_2(.|.), \delta_2, F_1(.|., .))$ with the required properties. Here, assume that ties are broken in favor of bidder 1. In Appendix 3, we explain the minor modifications a different tie-breaking rule requires. Because bidder 1's bid distribution is atomless in $\mathcal{E}$, the auction winner's revised beliefs are the same whether he wins with a bid strictly higher than or equal to his payment and hence are independent of his bid. We may thus consider simplified revised beliefs $F_j(.|.)$ that depend only on the payment. Since bidder i will be willing to offer a resale price that is also independent of his bid, we construct in $\mathcal{E}'$ resale-offer functions that are simplified functions[6] $\delta_i(v_i; b)$. As we show in Appendix 2, it is possible to construct $\mathcal{E}'$ in the following four steps.

<u>Step 1:</u> Construction of the supports: If v_i is in $[c'_i, d_i]$ and such that $\beta_i(v_i) < v_i$, the support of $G_i(.|v_i)$ is the interval of optimal bids in Result 1 (ii). Otherwise, the support of $G_i(.|v_i)$ is $\{\beta_i(v_i)\}$.

<u>Step 2:</u> Construction of revised beliefs $F_1(.|.)$ and $F_2(.|.)$ that are consistent with the supports in Step 1 and such that, when advantageous resale is possible, the auction winner finds it optimal to offer the same resale price

[5] However, bidder i might regret his entry decision. Notice that learning the payment in case of winning and learning the other bidder's bid are equivalent when $c'_2 = c''_2$, since then bids are different from the reserve price with probability one.

[6] Such a resale-offer function is formally defined as in Definition 1 (ii). However, here the second argument of δ_i is bidder i's payment and not his bid.

as in $\mathcal{E}$.

Step 3: Construction of the bidding strategy $G_i(.|.)$ as the conditional distribution of the bid with respect to the use value from the joint distribution of the use value-payment couples generated by the marginal $F_i\alpha_i$ over $[r, +\infty)$ of bidder i's payment in $\mathcal{E}$ and the conditional $F_i(.|.)$ from Step 2.

Step 4: Extension of the construction of optimal regular resale-offer functions from the domains in Step 2, where resale is possible, to the whole definition domain such that the resale offer does not depend on the bid from the auction loser along the equilibrium path.

If, as it is common, we interpret a bidder's acceptable bid as the price at which he exits the auction, $\mathcal{E}'$ in Result 2 also defines an equilibrium of the English auction, with initial entrance fees, no information release about the number of active bidders, and where the price starts rising from the reserve price.

4. Optimality of the Second-Price and English Auctions

Assume that the initial seller has some use value v_0 for the item and that the virtual-value functions are strictly increasing. From Myerson (1981), the seller maximizes his expected payoff if he keeps the item when no bidder has a virtual value larger than v_0, sells it to the bidder with the highest virtual use value otherwise, and leaves no payoff to the bidders with the smallest possible use values. We make the following "nondegeneracy assumption" which rules out those trivial cases where one bidder's virtual value is never larger than v_0 or than the other bidder's virtual value[7]:

$$\min(d_1, d_2) > \max(v_0, \omega_1(c_1), \omega_2(c_2)).$$

[7] If, for example, $d_2 \leq v_0$ or $d_2 \leq \omega_1(c_1)$, the optimal allocation is implemented by the second-price auction where the reserve price and entrance fees are such that only bidder 1 with virtual value larger than v_0 takes part (or, equivalently, a take-it-or-leave-it-offer to bidder 1). There will be no profitable resale to bidder 2.

Then, the "screening levels", that is, the smallest use values c_1', c_2' at which it can be optimal to allocate the item are:

$$c_i' = \omega_i^{-1} \left(\max \left(v_0, \omega_1 \left(c_1 \right), \omega_2 \left(c_2 \right) \right) \right), \ (3)$$

for all $i = 1, 2$. Without loss of generality, we may assume $c_1' \leq c_2'$.

We may define the function ψ that determines the optimal allocation as follows:

$$\psi \left(v_1 \right) = \omega_2^{-1} \left(\min \left(d_2, \omega_1 \left(\max \left(v_1, c_1' \right) \right) \right) \right),$$

for all v_1 in $[c_1, d_1]$. Any optimal mechanism allocates the item to bidder 2 if $v_2 > \psi \left(v_1 \right)$ and to bidder 1 if $v_1 > c_1'$ and $v_2 < \psi \left(v_1 \right)$.

If there exists a strictly increasing and continuous function φ^* such that $\varphi^* \left(c_1' \right) = c_2'$, $\varphi^* \left(d_1 \right) = d_2$, and $\lambda^{\varphi^*} = \psi$, where λ^{φ^*} is as defined in Result 1 (iii) (Section 3.1), then the regular equilibrium of the second-price auction constructed from φ^* will implement the optimal allocation. In fact, according to Result 1 (iii) (Section 3.1), it will allocate the item according to $\lambda^{\varphi^*} = \psi$. Assumption A below guarantees the existence of such a function φ^*.

(Weak) Assumption A: *ω_1 and ω_2 are strictly increasing and:*

(i) The unique continuous function μ_2 defined over $C = \{v_1 \in (c_1', d_1] \, | \psi \left(v_1 \right) \geq v_1\}$ and such that $\mu_2 \left(v_1 \right) \geq \psi \left(v_1 \right)$ and $\rho \left(v_1, \mu_2 \left(v_1 \right) \right) = \psi \left(v_1 \right)$, for all v_1 in C, is (nondecreasing) strictly increasing.

(ii) The unique continuous function μ_1 defined over $D = \left\{ v_2 \in (c_2', \rho \left(\min \left(d_1, d_2 \right), d_2 \right)] \, | \psi^{-1} \left(v_2 \right) \geq v_2 \right\}$ and such that $\mu_1 \left(v_2 \right) \geq \psi^{-1} \left(v_2 \right)$ and $\rho \left(\mu_1 \left(v_2 \right), v_2 \right) = \psi^{-1} \left(v_2 \right)$, for all v_2 in D, is (nondecreasing) strictly increasing[8].

The existence of the functions μ_1, μ_2 as defined above comes from the continuity and, when different from the lower extremities of the supports, the strict monotonicity of ρ. When Assumption A is satisfied, we simply

[8] It is straightforward to show that the continuous extension of the inverse ψ^{-1} of ψ is uniquely determined over $[c_2', \rho \left(\min \left(d_1, d_2 \right), d_2 \right)]$ and that $\psi^{-1} \left(c_2' \right) = c_1'$ and $\psi^{-1} \left(\rho \left(\min \left(d_1, d_2 \right), d_2 \right) \right) = \rho \left(d_1, \min \left(d_1, d_2 \right) \right)$.

construct φ^* from μ_1 and μ_2 as follows: $\varphi^* = \mu_2$ over C and $\varphi^* = \mu_1^{-1}$ over $\mu_1(D)$. If, given a reserve price $r^* \leq c_1'$, the entrance fees are equal to their maximum values $e_1(r^*)$ and $e_2(r^*)$, defined in Result 1 (Section 3.1), the bidders' equilibrium payoffs vanish at the screening levels c_1', c_2', and Theorem 1 below then follows from Myerson (1981).

Theorem 1: *Let Assumption A be satisfied. Let $r^*, e_1(r^*), e_2^*(r^*)$ be as in Result 1 (Section 3.1) where c_1', c_2' are the screening levels (9) with $c_1' \leq c_2'$ and $c_2'' = \mu_2(c_1')$. Then, the second-price and English auctions with reserve price r^*, entrance fees $e_1(r^*), e_2(r^*)$, and arbitrary tie breaking rule have a regular equilibrium that satisfies the no-regret property (defined in Result 2 (iii), Section 3.2) and that is optimal among all incentive-compatible and individually rational mechanisms.*

Notice that if the screening levels c_1', c_2' are identical, the reserve price $r^* = c_1'$ suffices, since then $e_i(r^*) = 0$, for all i. From Result 1 (Section 3.1), optimality is also achieved by a regular equilibrium of the sealed-bid auction with deferred payment, the same reserve price and entrance fees, and where bidder 1 wins ties.

If we require, as in the Weak Assumption A, the functions μ_1 and μ_2 to be only nondecreasing, the same construction as above will produce a nondecreasing function φ^* with, possibly, discontinuity jumps under the 45-degree line and flat portions above the 45-degree line. If we allow in our definitions bidding functions with discontinuity jumps above the reserve price and bidding strategies mixing over nonconvex supports, our results extend straightforwardly to this case. Under Zheng (2002)'s "Uniform Bias Assumption," according to which the use value supports are nested and the hazard rates are ranked, that is, $[c_1, d_1] \subseteq [c_2, d_2]$ and $\omega_1(.) \geq \omega_2(.)$ (and, consequently, $c_1' \leq c_2'$ and $C = (c_1', d_1])$, our Weak Assumption A reduces to his "Resale Monotonicity Assumption[9]."

[9] Zheng (2002) uses the notation β_{12} for our φ^*. A referee summarized our extension of the construction of φ^* as follows: "The idea is to observe, from the continuity of the Myerson allocation, that if the Myerson allocation favors bidder i against j at x_i, then

In Appendix 4, we explicitly work out one of Zheng (2002)'s examples.

5. Back to Zheng's Mechanism.

Zheng's mechanism also uses φ^* from our previous section to determine the mechanism "winner," to whom it awards the item, from the use values v_1, v_2 the bidders provide as inputs (see footnote 9). Under Zheng (2002)'s assumptions, $\varphi^*(v_1)$ is never below v_1 and hence only bidder 1 can be a reseller at equilibrium. When he is awarded the item, bidder 1 pays a price that depends only on his own input and such that his total expected payoff is equal to his expected payoff, conditional on the intermediate allocation, from Myerson's optimal mechanism. When bidder 2 is awarded the item, he pays the smallest use value at which, given bider 1's use value, Myerson's mechanism would have awarded him the item.

From our results, Zheng's mechanism can easily be amended to accommodate those cases that satisfy the Weak Assumption A (Section 4) and where hazard rates are not ranked. In such a case, the function φ^* can cross the 45-degree line and bidders 1 and 2 can be resellers at equilibrium. The mechanism may simply require from each bidder a payment equal to his total expected payment, including the entrance fee and conditional on his use value and the identity of the winner, at our optimal equilibrium of the second-price or English auction. As for bidder 1 in Zheng's original mechanism, a bidder's payment then depends only on his own input and is such that his total expected payoff, which includes his payoff from the resale stage, is equal to his expected payoff from Myerson's mechanism, conditional on the intermediate allocation.

A winner of this amended mechanism learns the same information as in the optimal equilibrium of the sealed-bid auction and hence will propose

it continues to favor i against j at any x_i' sufficiently close to x_i. Then within this neighborhood of x_i one can derive a winner-selection rule β_{ij} as β_{12}. Then do that for all neighborhoods, within each of which the relationship of favoring one against the other is stable. Then patch up all the winner-selection rules β_{ij} obtained locally, and we obtain a global winner-selection rule, ..."

the same resale price. Truth-telling is an equilibrium of Zheng's amended mechanism, otherwise bidders would deviate from their equilibrium strategies in the second-price auction.

With n bidders, Zheng's mechanism can be amended to fit cases that satisfy all his assumptions with the exception that the hazard rates of bidders $n-1$ and n may not be ranked between them. In fact, the mechanism looks for a winner by inspecting bidders in increasing index order. Once it reaches bidder $n-1$, it reduces to a two-bidder mechanism, which we can then amend as above. This change is inconsequential to the behaviors of bidders at earlier stages.

6. Conclusion.

We showed that Zheng (2002)'s optimal mechanism in the presence of resale can be interpreted as an equilibrium of an ascending-price auction and, in the two-bidder case, as an equilibrium with a no-regret property of the English and second-price auctions. It is somewhat surprising that such a novel and apparently complex mechanism actually describes an equilibrium of more familiar auctions. We also showed that it can be extended beyond Zheng (2002)'s original assumptions.

Appendix 1: Outline of the Proof of Result 1 (Section 3.1)

Assume that β_1, β_2 are the regular bidding functions the bidders are expected to follow at auction. Assume bidder i is declared the winner in any tie that may occur with strictly positive probability. Bidder i's updated beliefs about bidder j's use value after winning the auction with a bid b_i are represented by the conditional of F_j on $[c_j, \alpha_j(b_i)]$. Then, as stated in the main text, $\gamma_i(v_i; b_i) = \rho_i(v_i, \max(v_i, \alpha_j(b_i)))$ in Result 1 is the smallest resale price that maximizes his expected payoff.

Assume that the bidders choose their resale prices according to these resale-offer functions. When looking for regular equilibria, we may focus on the difference between bidder i's expected utility and his expected utility from losing with probability one. In fact, his utility when losing u_i^l does not depend on his bid. Then, bidder i's net value u_i for winning, that is, the difference between his utility when winning (gross of the auction price) u_i^w and his utility when losing u_i^l is as follows.:

$$\begin{aligned}
& u_i\left(v_i, v_j; b_i, b_j; \beta_i, \beta_j\right) \\
= \;& \rho_i\left(v_i, \max\left(v_i, \alpha_j\left(b_i\right)\right)\right), \text{ if not larger than } v_j; \\
= \;& \rho_j\left(v_j, \max\left(v_j, \alpha_i\left(b_j\right)\right)\right), \text{ if not larger than } v_i; \\
= \;& v_i, \text{ otherwise;}
\end{aligned}$$

for all couple of bids (b_1, b_2) in $(\{-1\} \cup [r, +\infty))^2$, couple of use values (v_1, v_2) in $[c_1, d_1] \times [c_2, d_2]$, and i, j such that $\{i, j\} = \{1, 2\}$.

Since bidder i's bid can enter his net value only as an argument of his resale price, which, we have assumed, he chooses optimally, $b_i' = b_i$ is a solution of the maximization problem below:

$$b_i \in \arg\max_{b_i' \geq r} \int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; b_i', \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right). \quad \text{(A1.1)}$$

By, as in Lebrun (2006), applying an envelope theorem, we find the first property below, which allows to circumvent the direct dependence of u_i on the own bid b_i. The other properties also mainly follow from Lebrun (2006)[10].

Properties of the Net-Value Functions: Assume $\beta_2\left(c_2''\right) \geq \beta_1\left(c_1'\right)$ and $\beta_1\left(c_1''\right) \geq \beta_2\left(c_2'\right)$, where c_i', c_i'' and β_i are as in Definition 1 (i), for all $i = 1, 2$.

[10] Below, we use (iii.2) to prove the optimality of the participation decisions. Lebrun (2006) did not need this property because he assumed mandatory participation.

(i) Envelope Property: For all v_i in $[c_i, d_i]$ and all b_i in $\left[\beta_j\left(c_j'\right), +\infty\right)$:

$$\begin{aligned}&\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; b_i, \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right)\\ = &\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; \max\left(\beta_j\left(v_j\right), \beta_j\left(c_j'\right)\right), \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right).\end{aligned}$$

(ii) Common Value for Pivotal Bids when Bidding as Expected: For all b in $(\max_i \beta_i\left(c_i''\right), \min_i \beta_i\left(d_i\right))$:

$$u_1\left(\alpha_1\left(b\right), \alpha_2\left(b\right); b, b; \beta_1, \beta_2\right) = u_1\left(\alpha_1\left(b\right), \alpha_2\left(b\right); b, b; \beta_1, \beta_2\right) = \rho\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right).$$

(iii) Monotonicity with Respect to Own Type: For all b in $(\max_i \beta_i\left(c_i''\right), \min_i \beta_i\left(d_i\right))$ and all $b_i \geq r$:

(iii.1) $u_i\left(v_i, \alpha_j\left(b\right); b, b; \beta_i, \beta_j\right)$ is nondecreasing with respect to v_i in $[c_i, d_i]$,

(iii.2) $\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; b_i, \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right)$ is nondecreasing with respect to v_i in $[c_i, d_i]$.

Proof: We amend the proof of (i) in Lebrun (2006) to take into account the possible jump of β_j at c_j''. From (2), the integral $\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; b_i, \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right)$ is constant in b_i over $\left[\beta_j\left(c_j'\right), \beta_j\left(c_j''\right)\right]$ and the equality in (i) holds true over this interval.

We next observe that, for all v_i in $[c_i, d_i]$ and almost all v_j in $[c_j, d_j]$, $u_i\left(v_i, v_j; b_i, \beta_j\left(v_j\right); \beta_i, \beta_j\right)$ is right-continuous with respect to b_i at $b_i = \beta_j\left(c_j''\right)$. In fact, from its definition and the continuity from the right of α_j, it is continuous from the right with respect to b_i at $b_i = \beta_j\left(c_j''\right)$, for all $v_j \neq \rho_i\left(v_i, \max\left(v_i, c_j''\right)\right)$. Applying the envelope theorem as in Lebrun (2006) and using this right-continuity, we find that the expected net value is equal to $\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; \max\left(\beta_j\left(v_j\right), \beta_j\left(c_j''\right)\right), \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right)$, for b_i in $\left[\beta_j\left(c_j''\right), \beta_j\left(d_j\right)\right)$. From the previous paragraph, the equality in (i) follows.

The equality in (i) over the interval $\left[\beta_j\left(d_j\right), +\infty\right)$ is then proved as in

Lebrun (2006) (by reasoning as in the first paragraph above).

(ii) and (iii.1) are proved as in Lebrun (2006).

Bidder i obtains the net expected payoff $\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; b_i, \beta_j\left(v_j\right); \beta_i, \beta_j\right) dF_j\left(v_j\right)$ if he proposes his optimal resale price $\rho_i\left(v_i, \max\left(v_i, \alpha_j\left(b_i\right)\right)\right)$ when he wins. It is thus the maximum of the net expected payoff he obtains when he proposes p_i, over all possible resale prices p_i. Since, for any fixed p_i, his net expected payoff is nondecreasing in his use value v_i, so will his optimal net expected payoff. (iii.2) is proved. ||

Result 1 (i) is an easy consequence of the properties above. In fact, bidder i's expected net payoff when his use value is v_i and his bid b is in $\left(\max_i \beta_i\left(c_i''\right), \min_i \beta_i\left(d_i\right)\right)$ is as follows:

$$\int_{c_j}^{\alpha_j(b_i)} u_i\left(v_i, v_j; \max\left(\beta_j\left(v_j\right), \beta_j\left(c_j'\right)\right), \beta_j\left(v_j\right); \beta_i, \beta_j\right) - \max\left(\beta_j\left(v_j\right), r\right) dF_j\left(v_j\right).$$

Since, at an equilibrium, b should be optimal for $v_i = \alpha_i(b)$, we obtain the following first-order condition:

$$\rho\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right) = b.$$

From Property (iii.1), the "second-order" condition is satisfied and any bid the bidding function specifies at or under the reserve price is optimal in this range. Bidder i's entrance fee e_i is chosen in Result 1 such that it is not larger than bidder i's expected net payoff at c_i' and equal to it when $c_i' > c_i$. From Property (iii.2), the participation decision is optimal and any couple of strategies as in Result 1 form an equilibrium.

As in Lebrun (2006), from the first-order condition and the envelope property we obtain the sets of optimal bids in Result 1 (ii). Result 1 (iii) can be proved as indicated in the main text.

Appendix 2: Outline of the Proof of Result 2 (Section 3.2).

Step 1 of the construction of $\mathcal{E}'$ from $\mathcal{E}$ is a simple definition. To show how Step 2 can be carried out, consider, for example, b in $(c_2', \rho(d_1, d_2))$ such that $\alpha_1(b) < \alpha_2(b)$. From Step 1, the support of $F_2(.|b)$ is equal to $\left[b, \varphi_2^+(b)\right]$, where $\varphi_2^+(b) = \min(\{d_2\} \cup \{w \in [c_1'', d_1] \,|\varphi(w) = w \geq b\})$. For all w in this interval and not larger than $\rho(d_1, d_2)$, in order for bidder 1 with use value $\alpha_1(w)$ to propose the same price w he would propose in $\mathcal{E}$, w must maximize bidder 1's expected payoff at resale. Integrating the first-order condition[11] of this maximization problem in w from b to v_2 in $\left[b, \min\left(\varphi_2^+(b), \rho(d_1, d_2)\right)\right]$ determines $F_2(.|b)$ over this interval. Because no resale price larger than $\rho(d_1, d_2)$ is proposed along the equilibrium path of $\mathcal{E}$, bidder 1's revised beliefs are not uniquely determined over $\left(\rho(d_1, d_2), \varphi_2^+(b)\right)$ when $\rho(d_1, d_2) < \varphi_2^+(b)$ and thus $d_1 < \varphi_2^+(b) = d_2$. In this case, an example of equilibrium beliefs is obtained by extending the first-order condition and, by integration, the revised beliefs to all w and v_2 in $\left[b, \varphi_2^+(b)\right]$ as follows:

$$\begin{aligned}(w - \widetilde{\alpha}_1(w)) f_2(w|b) &= 1 - F_2(w|b) \\ F_2(v_2|b) &= 1 - \exp \int_b^{v_2} \frac{1}{\widetilde{\alpha}_1(w) - w} dw, \text{ (A2.1)}\end{aligned}$$

where $\widetilde{\alpha}_1$ is the continuous extension of α_1 over $[r, d_2]$ such that $\rho(\widetilde{\alpha}_1(b), d_2) = b$, for all b in $[\rho(d_1, d_2), d_2]$. Formula (A2.1) indeed defines a probability distribution with the specified support because its R.H.S. tends towards one as v_2 tends towards $\varphi_2^+(b)$ (for a proof, see Lebrun, 2006)[12].

In the general case, for the purpose of defining the revised beliefs and, as we show below, the strategies, the function α_i, for all $i = 1, 2$, is extended as a continuous and nondecreasing function $\widetilde{\alpha}_i$ over $[r, \max(d_1, d_2)]$

[11] This necessary first-order condition will actually be sufficient and w will be the optimal resale offer for bidder 1 with use value $\alpha_1(w)$.

[12] When the payment b is the reserve price, the revised beliefs must account for the possibility that bidder 2 did not take part in the auction. If $F_2(c_2'') > 0$, we have $F_2(v_2|b) = F_2(v_2)/F_2(c_2'')$, for all v_2 in $[c_2, c_2']$, and $F_2(v_2|b) = 1 - \frac{F_2(c_2'') - F_2(c_2')}{F_2(c_2'')} \exp \int_{c_2'}^{v_2} \frac{1}{\alpha_1(w) - w} dw$, for all v_2 in $\left[c_2', \varphi_2^+(r)\right]$.

into $[c_i', \max(d_1, d_2)]$ such that:

$$\rho(\widetilde{\alpha}_1(w), \widetilde{\alpha}_2(w)) = w, \text{ (A2.2)}$$

for all w in $[r, \max(d_1, d_2)]$. From Result 1 (i), such extensions are possible. As the example above where $d_1 < d_2$ shows, only at most one function among α_1, α_2 needs to be extended strictly.

Step 3 leads to a bidding strategy of bidder i if and only if the marginal distribution F_i^* of the joint distribution generated by $F_i\alpha_i$ (over $[r, +\infty)$) and $F_i(.|.)$ is equal over $[c_i', d_i]$ to the actual distribution F_i of bidder i's use value. To show that this is indeed the case, take, for example, v in $[c_2', d_2]$ such that $\beta_2(v) < v$. Then, for all w in $\left(\varphi_2^-(v), \varphi_2^+(v)\right)$, where $\varphi_2^-(v) = \max(\{c_2'\} \cup \{w \in [c_1'', d_1] \,|\varphi(w) = w \leq v\})$, and all b in $\left[\beta_2\left(\varphi_2^-(v)\right), \min(w, \rho(d_1, d_2))\right]$, the first-order condition (A2.1) holds true. Integrating it with respect to the payment b, we find that F_2^* satisfies over the interval $\left(\varphi_2^-(v), \varphi_2^+(v)\right)$ the same differential equation that, from (A2.2), F_2 satisfies. Because these two cumulative functions coincide at the extremities of this interval, they are identical everywhere inside it (for more details, see Lebrun 2006)[13].

Step 4 can be carried out by defining the following resale-offer function, for all (v_i, b) in $[c_i, d_i] \times [r, +\infty)$:

$$\begin{aligned}\delta_i(v_i, b) &= \rho_i\left(v_i, \max\left(c_j'', v_i\right)\right), \text{ if } v_i \in [c_i, c_i') \text{ and } b = r;\\ &= \max(v_i, d_j), \text{ if } b > \rho(d_1, d_2);\\ &= \max(\min(b, \alpha_j(b)), \max(\beta_i(v_i), v_i)), \text{ otherwise.}\end{aligned}$$

[13]For b' in $\left[\beta_2\left(\varphi_2^-(v)\right), \min(v_2, \rho(d_1, d_2))\right]$, the explicit formula for $1 - G_2(b'|v_2)$ can be obtained by differentiating (A2.1) with respect to v_2, integrating with respect to b from b' to $\min(v_2, \rho(d_1, d_2))$ according to $F_2\alpha_2$, and dividing by $f_2(v_2)$. Proceeding in this way, we find:

$$G_2(b'|v_2) = 1 - \frac{\int_{b'}^{\min(v_2, \rho(d_1, d_2))} \exp \int_b^{v_2} \frac{dw}{\widetilde{\alpha}_1(w) - w} dF_2\alpha_2(b)}{f_2(v_2)(v_2 - \widetilde{\alpha}_1(v_2))}.$$

If we then define as in Appendix 1 the net-value functions u_1, u_2 from these resale-offer functions, we obtain the properties below. Because we only consider β_i, β_j from $\mathcal{E}$, we drop them from the argument of u_i. Notice the change of lower extremity in (i) with respect to the similar property in Appendix 1. Since here the utility in case of winning does not depend on the own bid, we compare the expected utility to the expected utility from winning with probability one, that is, for all use values of the opponent.

Properties of the Net-Value Functions:

(i) For all v_i in $[c_i, d_i]$ and all b_i in $[r, +\infty)$:

(i.1)

$$\begin{aligned} & \int u_i\left(v_i, v_j; b_i, b\right) dF_j\left(v_j|b\right) \\ = & \int u_i\left(v_i, v_j; b, b\right) dF_j\left(v_j|b\right), \end{aligned}$$

for all $b \geq b_i$, and

(i.2)

$$\begin{aligned} & \int_{\rho(d_1,d_2)}^{b_i} \int u_i\left(v_i, v_j; b_i, b\right) dF_j\left(v_j|b\right) dF_j\alpha_j\left(b\right) \\ = & \int_{\rho(d_1,d_2)}^{b_i} \int u_i\left(v_i, v_j; b, b\right) dF_j\left(v_j|b\right) dF_j\alpha_j\left(b\right). \end{aligned}$$

(ii) For all $b > c_2'$:

$$\int u_i\left(\alpha_i\left(b\right), v_j; b, b\right) dF_j\left(v_j|b\right) = \rho\left(\alpha_1\left(b\right), \alpha_2\left(b\right)\right).$$

(iii) For all $b \geq r$, $\int u_i\left(v_i, v_j; b, b\right) dF_j\left(v_j|b\right)$ is nondecreasing with respect to v_i in $[c_i, d_i]$.

Proof: (i.1) is immediate since, from Step 4, the net-value of the auction loser at resale does not depend on his bid. (i.2) follows from (i.1). (ii) holds true because resale occurs with probability one when both bidders submit the same bid b (and $\alpha_1(b) \neq \alpha_2(b)$). If, for example, $\alpha_2(b) > \alpha_1(b)$, bidder 1's use value is $\alpha_1(b)$ and bidder 1's resale offer $\rho(\alpha_1(b), \alpha_2(b))$ is accepted with probability one by bidder 2, since it is the minimum of the revised support of his use value. (iii) can be proved as (iii.2) in Appendix 1. ||

Proceeding as in Appendix 1, we obtain in $\mathcal{E}'$ the same sets of optimal bids for the use values leading to participation in the auction in $\mathcal{E}$. Since those sets are the supports of the bidding strategies $G_i(.|.)$, $i = 1, 2$, bidders in $\mathcal{E}'$, when they take part in the auction, submit optimal bids.

The participation decisions are also optimal. If bidder i's use value is c_i', his expected payoff, gross of the entrance fee, when he submits r, which is optimal in $[r, +\infty)$, is equal to[14] $e_i(r)$. Since bidder i can always replicate what he does for a smaller use value and obtain a payoff at least as high, the decision to participate only when his use value is at least c_i' is optimal. Consequently, $\mathcal{E}'$ is a regular equilibrium.

From (i) and (ii) (and (1)), when a bidder takes part, according to $\mathcal{E}'$, in the auction, any of his equilibrium bids wins against bids that would contribute nonegatively to his net expected payoff and loses against those that would contribute nonpositively. The no-regret property follows.

To check that the final allocation is the same in $\mathcal{E}'$ as in $\mathcal{E}$, assume, for example, that bidder 1's use value v_1 is such that $v_1 \geq c_1'$ and $\varphi(v_1) \geq v_1$. Then, bidder 1 bids $\beta_1(v_1)$ and we have $v_1 \leq \lambda^{\varphi}(v_1) = \rho(v_1, \varphi(v_1)) \leq \varphi(v_1)$. If $v_2 \leq \lambda^{\varphi}(v_1)$, Step 2 implies that bidder 2 with use value v_2 does not bid higher than $\beta_1(v_1)$. Thus, bidder 2 loses the auction and refuses bidder 1's offer. If $v_2 \geq \lambda^{\varphi}(v_1)$, bidder 2 accepts bidder 1's resale offer when bidder 1 wins and there is no profitable resale if bidder 2 wins[15]. The rest of Result

[14] In fact, when the bid c_2' of bidder 1 with use value c_1' wins, which occurs with probability $F_2(c_2'')$, he offers the resale price c_2', which bidder 2 refuses with conditional probability $F_2(c_2')/F_2(c_2'')$. Bidder 1's expected payoff is then $e_1(r)$. Since bidder 1 offers at least c_2' as a resale price and since no resale is possible when bidder 2 with use value c_2' and bid r wins, such a bidder 2's expected payoff is $e_2(r)$.

[15] As an example of a use value such that the bidder stays out of the auction, consider

2 then follows from Myerson (1981).

Appendix 3

If the tie-breaking rule chooses bidder 1 with probability $q \neq 1$, only the following specifications differ:

$$G_1\left(.|c_1'\right) \text{ is concentrated at } -1.$$

$$\begin{aligned} F_2\left(v_2|r,r\right) &= \frac{F_2\left(v_2\right)}{\left(1-q\right)F_2\left(c_2'\right)+F_2\left(c_2''\right)}, \text{ if } v_2 \in \left[c_2, c_2'\right] \\ &= 1-\frac{q\left(F_2\left(c_2''\right)-F_2\left(c_2'\right)\right)}{\left(1-q\right)F_2\left(c_2'\right)+F_2\left(c_2''\right)} \exp\int_{c_2'}^{v_2}\frac{dw}{\alpha_1\left(w\right)-w}, \text{ if } v_2 \in \left[c_2', \varphi_2^+\left(r\right)\right]. \end{aligned}$$

$$\delta_1\left(v_1;r,r\right) = \arg\max_{p \geq v_1}\left(v_1-p\right)\left(1-F_2\left(p|r,r\right)\right).$$

Otherwise, the simplification in the text and in Appendix 2, according to which the revised beliefs and the resale offers do not depend on the own bid, still applies. Since bidder 1's acceptable bid distribution is atomless, nothing is changed for bidder 2. Bidding the reserve price is the only deviation that the change of tie-breaking rule may render profitable for bidder 1. However, for bidder 1 with use value at least c_1', bidding slightly above the reserve price is more advantageous because it increases the probability of obtaining the item at a price–r–not larger than the use value (here we use $r \leq c_1'$) and than any resale price bidder 2 may offer when he bids r. Moreover, when bidder 1's use value is c_1', bidding slightly above r gives bidder 1 a gross expected payoff equal to $e_1\left(r\right)$. Reasoning as in Appendix 2, it is optimal for bidder 1 with use value $v_1 < c_1'$ not to take part.

Appendix 4

$v_1 < c_1'$. Bidder 2 takes part in the auction only when $v_2 \geq c_2'$, in which case, since $c_2' \geq c_1'$, no resale is possible.

Here, we work out Zheng (2002)'s example where F_1 and F_2 are uniform distributions over intervals $[c_1, d_1]$ and $[c_2, d_2]$ such that $v_0 \in [c_1, d_1] \subseteq [c_2, d_2]$. The virtual-value functions are $\omega_1(v_1) = 2v_1 - d_1$, $\omega_2(v_2) = 2v_2 - d_2$ and the screening levels are then $c_1' = \frac{v_0+d_1}{2} \leq c_2' = \frac{v_0+d_2}{2}$. The function ψ such that $\psi(v_1) = v_1 + \frac{d_2-d_1}{2}$, for all v_1 in $[c_1', d_1]$, determines the optimal final allocation. Since C in Assumption A (i) (Section 4) is the whole interval $(c_1', d_1]$, the function φ^* that determines the optimal intermediate allocation is equal to μ_2 and we have $\varphi^*(v_1) = v_1 + d_2 - d_1$, for all v_1 in $[c_1', d_1]$. The value $\frac{v_0-d_1}{2} + d_2$ for $c_2'' = \varphi^*(c_1')$ follows.

Let us set, for example, the reserve price r at c_1'. Then, from the formulas in Result 1 (Section 3.1), the entrance fees $e_1(r)$ and $e_2(r)$ are, respectively, $\left(\frac{d_2-d_1}{2}\right)^2 \frac{1}{d_2-c_2}$ and $\frac{d_2-d_1}{4}\left(1 + \frac{v_0-c_1}{d_1-c_1}\right)$. Again from Result 1, the following bidding functions form an optimal equilibrium of the second-price auction with deferred payment (see Figure 1):

$$\begin{aligned} \beta_1(v_1) &= -1, \text{ if } v_1 < c_1'; \\ &= v_1 + \frac{d_2-d_1}{2}, \text{ if } v_1 \geq c_1'; \end{aligned}$$

$$\begin{aligned} \beta_2(v_2) &= -1, \text{ if } v_2 < c_2'; \\ &= r, \text{ if } c_2' \leq v_2 < c_2''; \\ &= v_2 - \left(\frac{d_2-d_1}{2}\right), \text{ if } v \geq c_2''; \end{aligned}$$

The inverse bidding functions are such that $\alpha_1(b) = b - \left(\frac{d_2-d_1}{2}\right)$ and $\alpha_2(b) = b + \left(\frac{d_2-d_1}{2}\right)$, for all b in $\left[c_2', \frac{d_1+d_2}{2}\right]$.

FIGURE 1

Following the construction of the equivalent equilibrium in Result 2 (Section 3.2), bidder 1 follows the same bidding strategy[16] in the standard second-

[16] Except that, if the tie-breaking rule does not favor bidder 1, he does not take part in

price auction and bidder 2, when he takes part in the auction, randomizes over bids. As we explain in Appendix 2, in order to obtain formulas for bidder 1's revised beliefs and bidder 2's behavioral strategy, we can extend α_1 to the interval $[c_2', d_2]$ according to $\rho(\widetilde{\alpha}_1(b), d_2) = b$ or, equivalently, $\widetilde{\alpha}_1(b) = \omega_2(b) = 2b - d_2$, for all b in $\left[\frac{d_1+d_2}{2}, d_2\right]$. The formula (A2.1) then gives[17]:

$$
\begin{aligned}
F_2(v_2|b) &= 1 - \exp\frac{2(b - v_2)}{d_2 - d_1}, \text{ if } b \leq v_2 \leq \frac{d_1 + d_2}{2}; \\
&= 1 - \frac{2(d_2 - v_2)}{d_2 - d_1}\exp\frac{2b - d_1 - d_2}{d_2 - d_1}, \text{ if } \frac{d_1 + d_2}{2} \leq v_2 \leq d_2;
\end{aligned}
$$

for all b in $\left[c_2', \frac{d_1+d_2}{2}\right]$. Proceeding as in Step 3 (see Footnote 13, Appendix 2), we find:

$$G_2(b_2|v_2) = \exp\frac{2(b_2 - v_2)}{d_2 - d_1}, \text{ if } c_2' \leq b_2 \leq v_2;$$

for all v_2 in $\left[c_2', \frac{d_1+d_2}{2}\right]$, and

$$G_2(b_2|v_2) = \exp\frac{2b - d_1 - d_2}{d_2 - d_1}, \text{ if } c_2' \leq b_2 \leq \frac{d_1 + d_2}{2};$$

for all v_2 in $\left[\frac{d_1+d_2}{2}, d_2\right]$. In both cases, $G_2(.|v_2)$ is constant over $[r, c_2']$.

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the auction when his use value is c_1' (see Appendix 3).

[17] We assume that the beliefs for b in (r, c_2') are the same as for $b = c_2'$. If the tie breaking rule favors bidder 1, the beliefs are also the same at $b = r$. Otherwise, they must be obtained as in Appendix 3.

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Figure